"QCD and Standard Model"
Problem Set 2

1. Fermions as representations of the Lorentz group

We start from the the left and right two-components Weyl spinors

$$u_L^{\alpha}(x) = \begin{pmatrix} u_{L\,1}(x) \\ u_{L\,2}(x) \end{pmatrix} \in \tau_{\frac{1}{2}0} \,, \qquad u_{R\,\dot\alpha}(x) = \begin{pmatrix} u_{R\,1}(x) \\ u_{R\,2}(x) \end{pmatrix} \in \tau_{0\frac{1}{2}} \,. \tag{1}$$

For the sake of simplicity, we drop the spinorial indices α and $\dot\alpha$ in what follows. Under Lorentz transformations, the spinors behave as

$$u'_{L,R}(x') = S_{L,R}\, u_{L,R}(x) \,, \tag{2}$$

with the $SL(2,\mathbb{C})$ matrices

$$S_{L,R} = e^{-\frac{i\sigma_j}{2}(\theta_j \mp i\phi_j)} \,. \tag{3a}$$

Here, σ_j $(j = 1,2,3)$ are the Pauli matrices, while θ_j, ϕ_j are the angle and rapidity parameters of the Lorentz group, respectively.

a) Prove the following properties of the $SL(2,\mathbb{C})$ matrices :

$$S_L^{-1} = S_R^{\dagger} \,, \tag{4a}$$
$$\sigma_2 S_L \sigma_2 = S_R^{*} \tag{4b}$$
$$S_L^{T} = \sigma_2 S_L^{-1} \sigma_2 \,. \tag{4c}$$

Of course, there are 3 similar identities that one gets by swapping $L \leftrightarrow R$.

b) Use the relations (4) to prove that
— Any left-handed Weyl spinor u_L is such that $\sigma_2 u_L^* \in \tau_{0\frac{1}{2}}$;
— Adding another left-handed spinor v_L, we have $v_L^T \sigma_2 u_L \in \tau_{00}$, *i.e.*, it is a scalar ;
— $u_L^{\dagger}(x)\sigma_-^{\mu} u_L(x) \in \tau_{\frac{1}{2}\frac{1}{2}}$, where $\sigma_-^{\mu} = (I, -\sigma^j)$ and I is the 2×2 identity matrix.
How can we use these properties to guess the Lagrangian density $\mathcal{L}[u_L, u_R]$?

c) However $SL(2,\mathbb{C})$ matrices realise a double covering of the full Lorentz group and the two chiralities mix under parity transformations. One has to introduce the *Dirac bispinor*

$$\psi(x) = \begin{pmatrix} u_L(x) \\ u_R(x) \end{pmatrix} \,, \tag{5}$$

and the Dirac matrices in the so-called *Weyl* or *chiral basis*

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma_+^{\mu} \\ \sigma_-^{\mu} & 0 \end{pmatrix} \,, \qquad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \,, \tag{6}$$

with $\sigma_+^{\mu} = (I, \sigma^j)$. Prove that the transformation law for ψ reads

$$\psi'(x') = e^{i\theta_{\mu\nu}\Sigma^{\mu\nu}}\, \psi(x) \,, \tag{7}$$

if one introduces the spin tensor

$$\Sigma^{\mu\nu} = \frac{1}{4i}[\gamma^{\mu}, \gamma^{\nu}] \,. \tag{8}$$

Hint : You may find useful to employ the transformations (3) *together with the definition* (5)*, and to compute the total variation* $\Delta\psi(x) = \psi'(x') - \psi(x)$ *by considering relativistic boosts and rotations separately.*

d) Show that the Dirac matrices do realise a matrix representation of the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \qquad \{\gamma^\mu, \gamma_5\} = 0 . \tag{9}$$

Use it to prove the properties

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 , \qquad \gamma_5{}^\dagger = \gamma_5 . \tag{10}$$

e) By using the relations of the previous points and the adjoint spinor $\bar\psi = \psi^\dagger\gamma^0$, show that $\bar\psi\psi$ is a scalar, whereas $\bar\psi\gamma^\mu\psi$ transforms as a vector. Therefore, we can construct the renowned Dirac Lagrangian

$$\mathcal{L}_{\rm D}[\psi] = i\bar\psi(x)\gamma^\mu\partial_\mu\psi(x) - m\bar\psi(x)\psi(x) \tag{11}$$

for a free bispinor of mass m.

f) Prove that one can construct two projectors

$$L = \frac{1}{2}(1+\gamma_5) , \qquad R = \frac{1}{2}(1-\gamma_5) , \tag{12}$$

such that $L\,\psi = (u_L, 0)^T$ and $R\psi = (0, u_R)^T$.

g) Finally, remember that there is another kind of bispinor only by using either ψ_L or ψ_R, instead both of them. These are the known as Majorana bispinors. Construct the charge conjugation operator, by using the matrices (6), and show that these objects are self-conjugated.