“QCD and Standard Model”
Problem Set 11

Chiral anomaly

Anomalies arise when a symmetry of the classical theory is violated at the quantum level. First, we discuss this important phenomenon by playing with the global symmetries of the massless QED Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}\left(\partial_{\mu} - iQA_{\mu}\right)\psi \ . \tag{1}$$

a) Prove that the Lagrangian is invariant under the global vector and axial $U(1)$ transformations

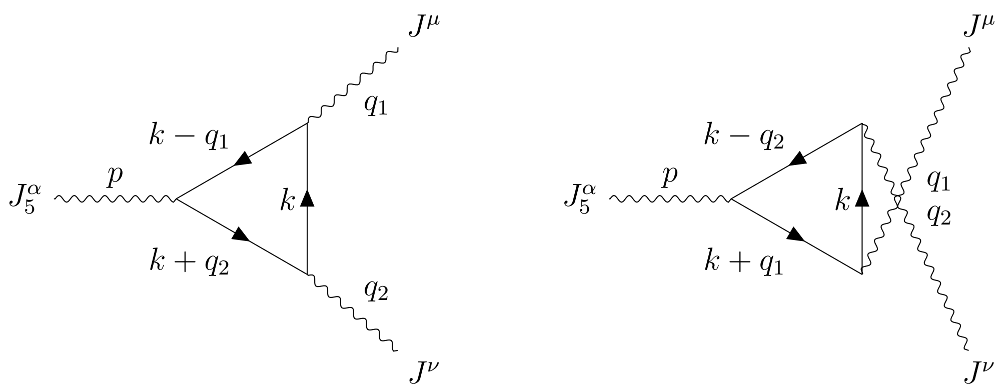
$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) \ , \qquad \psi(x) \to \psi'(x) = e^{i\beta\gamma^5}\psi(x) \tag{2}$$

and write down the corresponding Noether currents, called J^{μ} and J_5^{μ}, respectively. Show that the axial current J_5^{μ} is not conserved when one introduces a mass term for the fermion.

b) Compute the 3−point function (*Hint : Do not evaluate the integral, yet!*)

$$\mathcal{M}_5^{\alpha\mu\nu}(p,q_1,q_2)\delta^{(4)}(p-q_1-q_2) = -i\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\, e^{-ipx}e^{iq_1y}e^{iq_2z}\langle J_5^{\alpha}(x)J^{\mu}(y)J^{\nu}(z)\rangle \ ,$$

corresponding to the triangle diagrams



In the above wavy lines denote background photons, which can be coupled to either the vector or the axial current in the massless case.

Hint : Recall that the expression $\langle\ldots\rangle$ is just the vacuum expectation value of the time-ordered product of the currents, written as quantum operators.

Optional : Could you explain this identification between (unpolarised) photons and the insertion of momenta through currents?

c) Check the vector and axial Ward-Takahashi identities by computing the contractions $q^1_{\mu}\mathcal{M}_5^{\alpha\mu\nu}$ and $p_{\alpha}\mathcal{M}_5^{\alpha\mu\nu}$, respectively. In the vector case you should find a linearly divergent integral, which seems to vanish, if you perform the appropriate shift in the loop-momentum. This, however, is not correct. Show that for linearly divergent integrals, which would vanish for a certain loop-momentum shift, the result is finite and proportional to the necessary shift.

d) So it seems as if you just found that the Ward-Takahashi identity is violated for the vector current, but not for the axial current, which is the opposite of what we set out to show! The resolution is that although the choice of the loop-momentum is arbitrary, we have to stick to the same choice for all possible contractions of $\mathcal{M}_5^{\alpha\mu\nu}$. Now, let's use the most general possibility

$$k^\mu \to k^\mu + b_1 q_1^\mu + b_2 q_2^\mu \ , \tag{3}$$

for the first graph, and

$$k^\mu \to k^\mu + b_2 q_1^\mu + b_1 q_2^\mu \ , \tag{4}$$

for the second graph (why do we have to exchange b_1 and b_2 in the second graph?). Finally, calculate the Ward-Takahashi identities again and fix the ambiguity by requiring the conservation of the vector current.

e) Let us assume that the fermion content of the theory comprises a single left-handed Weyl spinor $\psi_L = P_L\psi = \frac{1}{2}(1+\gamma^5)\psi$ carrying a charge q_L and described by

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu\left(\partial_\mu - iQ_L A_\mu\right)P_L\psi \ . \tag{5}$$

The Lagrangian density is invariant under a global $U(1)_L$ symmetry with conserved current J_L^μ. Compute the amplitude $\mathcal{M}_L^{\alpha\mu\nu}$ from $\langle J_L^\alpha J_L^\mu J_L^\nu\rangle$ and check whether it is anomalous or not, following the steps above.

f) Include a massless right-handed fermion $\psi_R = P_R\psi$, with charge q_R, in the Lagrangian density (5). Now the Noether current reads $J^\mu_{\mathrm{mix}} = J_L^\mu + J_R^\mu$. Prove that

$$p_\alpha\mathcal{M}^{\alpha\mu\nu}_{\mathrm{mix}} \;=\; Q_L\, p_\alpha\mathcal{M}_L^{\alpha\mu\nu} + Q_R\, p_\alpha\mathcal{M}_R^{\alpha\mu\nu} \;=\; \frac{1}{2}\left(Q_R^3 - Q_L^3\right)p_\alpha\mathcal{M}_5^{\alpha\mu\nu} \ , \tag{6}$$

so the theory is anomaly free only if $Q_L = Q_R$, as in a Dirac spinor.