Übungen zu Theoretischer Mechanik (T1)

Blatt 11

1 Kepler und Himmelsmechanik

Im folgenden betrachten wir ein Teilchen der Masse m welches sich in einem Zentralpotintial $U(r)$ bewege. In der Vorlesung wurde gezeigt, dass der Drehimpuls $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ eine Erhaltungsgröße der Bewegung ist. In geeigneten Koordinten, lässt sich dieser durch den Winkel φ und den Radius r ausdrücken als

$$L_z =: L = mr^2\dot{\varphi}.$$

(i) Argumentieren Sie, weshalb die Bewegung immer innerhalb einer Ebene stattfindet.

Given a rotating particle of mass m, the reference frame can always be aligned with the plane of rotation, so that the rotating motion is always on a plane. In other words, one can align the reference frame such that $\boldsymbol{r} \times \boldsymbol{p} = L_z \hat{\boldsymbol{z}}$, which implies $\boldsymbol{p} \cdot \hat{\boldsymbol{z}}_z = 0$, indicating that $p_z = 0$.

(ii) Begründen Sie weshalb die Erhaltung von L impliziert, dass die Fläche welche der Radiusvektor in gegebener Zeit überstreicht konstant ist.

First of all, consider an infinitesimal area section,

$$dA(t) = \frac{1}{2} r^2 d\varphi(t)$$

and the areal velocity is,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\varphi}(t)$$

It also shows that the areal velocity is a factor of $1/2m$ compared with the angular momentum

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

Since L and m are constant, the areal velocity is constant. This implies equal area is sweeped by the radial vector at equal time:

$$A = \int_{t_0}^{t_1} \frac{dA}{dt}\, dt = \frac{L}{2m}(t_1 - t_0)$$

(iii) In Aufgabe 8.3 haben wir gezeigt, dass für ein Teilchen der Energie E der Winkel φ folgende Gleichung erfüllt

$$\varphi = \int dr \frac{\frac{L}{r^2}}{\sqrt{2m(E - U(r)) - \frac{L^2}{r^2}}} + \text{const.} \tag{1}$$

Zeigen Sie, dass die Bahngleichung für ein Potential der Form $U = -\alpha/r$ die Form eines Kegelschnittes annimmt

$$\frac{p}{r} = 1 + e\cos\varphi, \tag{2}$$

wobei p und e Konstanten sind die von L, m und E abhängen. Für welche E ist die Bahn Elliptisch?

Given the equation (1),

$$\varphi(t) - \varphi(t_0) = \int_{r(t_0)}^{r(t)} \frac{L^2}{r^2} \frac{dr}{\sqrt{2mE + (2m\alpha/r) - (L^2/r^2)}}$$

First of all, we consider a change of variable:

$$x = \frac{L}{r}, \qquad \frac{dx}{dr} = -\frac{L}{r^2}$$

which rearragne equation into the following,

$$\varphi(t) - \varphi(t_0) = -\int_{x(t_0)}^{x(t)} \frac{dx}{\sqrt{a + bx - x^2}}$$

and the coefficients are defined for convenient as follow,

$$a = 2mE, \qquad b = \frac{2m\alpha}{L},$$

Observe that,

$$a + bx - x^2 = d - \left(x - \frac{b}{2}\right)^2, \qquad d := a + \frac{b^2}{4}$$

so that,

$$\varphi(t) - \varphi(t_0) = -\int_{x(t_0)}^{x(t)} \frac{dx}{\sqrt{d - [x - (b/2)]^2}} = -\int_{u(t_0)}^{u(t)} \frac{du}{\sqrt{1 - u^2}}$$

Again, a new variable is defined,

$$u = \frac{x - (b/2)}{\sqrt{d}}, \qquad \frac{du}{dx} = \frac{1}{\sqrt{d}}$$

and

$$u = \cos\theta, \qquad \frac{du}{d\theta} = -\sin\theta$$

At last, we can integrate over the equation,

$$\varphi(t) - \varphi(t_0) = \int_{\theta_0}^{\theta} d\theta$$

This implies,

$$\theta = \varphi(t) - \tilde{\varphi}(t_0)$$

where integration constant is combined,

$$\tilde{\varphi}(t_0) = \varphi(t_0) + \theta_0$$

Now rewrite θ in terms of all the change of variables,

$$\cos(\varphi(t) - \varphi(t_0)) = \frac{(L/r) - (b/2)}{\sqrt{d}}$$

so we have,

$$\frac{2L}{b}\frac{1}{r} = 1 + \frac{2\sqrt{d}}{b}\cos(\varphi(t) - \varphi(t_0))$$

and we have recovered,

$$\frac{p}{r} = 1 + e\cos(\varphi(t) - \varphi(t_0))$$

where

$$p = \frac{2L}{b} = \frac{L^2}{m\alpha}, \qquad e = \frac{2\sqrt{d}}{b} = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}$$

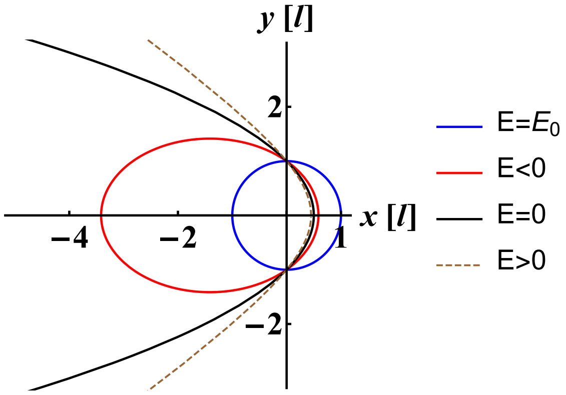

Abbildung 1: Trajectory of the particle with mass m with different energy E

From this expression, it shows that there exist four classes of trajectory that depends on the total energy E of the particle,

$$\begin{array}{lll} e=0, & E=E_0=-\dfrac{2EL^2}{m\alpha^2} & \text{Circular} \\ e<1, & E_0<E<0, & \text{Elliptic} \\ e=1, & E=0, & \text{Parabola} \\ e>1, & E>0, & \text{Hyperbola} \end{array}$$

See Figure 1 for the trajectory.

(iv) Welche Form haben Bahnen für die $E > 0$ ist?

For $E > 0$, the trajectory is hyperbolic.

Wir werden nun versuchen eine Parametrische Darstellung für die Zeitabhängigkeit der Koordinaten für den parabolischen Fall ($E = 0$) zu bestimmen.

(vi) Drücken Sie die Zeit t als Integral eine Funktion von r aus. Nutzen Sie die Substitution $r = \frac{p}{2}(1+\chi^2)$ um $t(\chi), x(\chi)$ und $y(\chi)$ zu bestimmen, wobei x und y die üblichen kartesischen Koordinaten in der Ebene sind.

In this section, we parametrize the function r, t, x, y with respect to the variable, χ. First of all, recall that,

$$\begin{aligned} t - t_0 &= \sqrt{\frac{m}{2}} \int_{r_0}^{r(t)} \frac{dr}{\sqrt{(E - U_{\text{eff}}(r))}} = \sqrt{\frac{m}{2}} \int_{r_0}^{r(t)} \frac{rdr}{\sqrt{(\alpha/r) - (L^2/(2mr^2))}} \\ &= \sqrt{\frac{m}{2\alpha}} \int_{r_0}^{r(t)} \frac{dr}{\sqrt{r-a}} \end{aligned}$$

where

$$a = \frac{L^2}{2m\alpha} = \frac{p}{2},$$

so that

$$t - t_0 = \sqrt{\frac{m}{2\alpha}} \int_{r_0}^{r(t)} \frac{rdr}{\sqrt{r-a}} = \sqrt{\frac{m(r-a)}{8\alpha}}$$

On the last step, we absorbed the integration constant into t_0. Now, the entire equation can be expressed as a function of χ defined as follow,

$$\chi = \sqrt{\frac{r-a}{a}}$$

In other words, the radial vector is parametrized by χ as follow,

$$r(\chi) = a + a\chi^2 = \frac{p}{2}\left(1 + \chi^2\right)$$

Such parametrization rewrite the function t as,

$$t(\chi) = t_0 + \frac{L^2}{4\alpha}\chi$$

In Cartesian coordinate system,

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

From our result in the last section, we have,

$$\cos\varphi = \frac{1}{e}\left(\frac{p}{r} - 1\right)$$

so that,

$$x(\chi) = \frac{p}{2e}\left(1 - \chi^2\right)$$

For $y^2 = r^2 - x^2$,

$$y(\chi) = \frac{p}{2e}\left[e^2(1+\chi^2)^2 - (1-\chi^2)^2\right]^{1/2}$$