Übungen zu Theoretischer Mechanik (T1)

Blatt 9

1 Fischen im Phasenfluss

In der Vorlesung wurde das Konzept des Phasenflusses eingeführt, als die Abbildung der Dynamik eines Systems auf den Phasenraum. Der Phasenraum einer gewöhnlichen Differenzialgleichung bezeichnet im Allgemeinen den Raum der möglichen Anfangswerte, in der Mechanik gleichzusetzen mit den Anfangs-Impulsen und Positionen. Die Dimension des Phasenraums hängt daher im Allgemeinen von der Zahl der dynamischen Gleichungen sowie ihrer Ordnung ab. Oftmals können qualitative Aspekte leicht anhand des Phasenflusses abgelesen werden. Wir betrachten als erstes ein einfaches Modell, zur Beschreibung der Karpfenpopulation in einem Teich. Die Anzahl der im Teich lebenden Karpfen zum Zeitpunkt t sei durch die Funktion $N(t)$ gegeben, welche der Gleichung

$$\dot{N} = (m - N)N \tag{1}$$

genügt, wobei $m \in \mathbb{N}$ ist.

(i) Welche Dimension hat der zu (1) gehörige Phasenraum?

Der zur Gleichung (1) gehörige Phasenraum hat eine Dimension. Ein mehrdimensionaler Phasenraum des gleichen Systems wäre eine Betrachtung mehrerer Populationen in einem Teich.

The phase space belonging to equation (1) has one dimension. A multi-dimensional phase space of the same system would be a consideration of several populations in a pond.

(ii) Zeichnen Sie die Integralkurven der Gleichung (1) und argumentieren Sie, dass das System Zwei Gleichgewichtslagen hat. Was ist die maximal mögliche Stabile Karpfenpopulation?

Zunächst zeichnen wir die Integralkurven für der Gleichung (1) für einen festen Wert $m = 1$.

First we draw the integral curves for equation (1) with a fixed value $m = 1$.

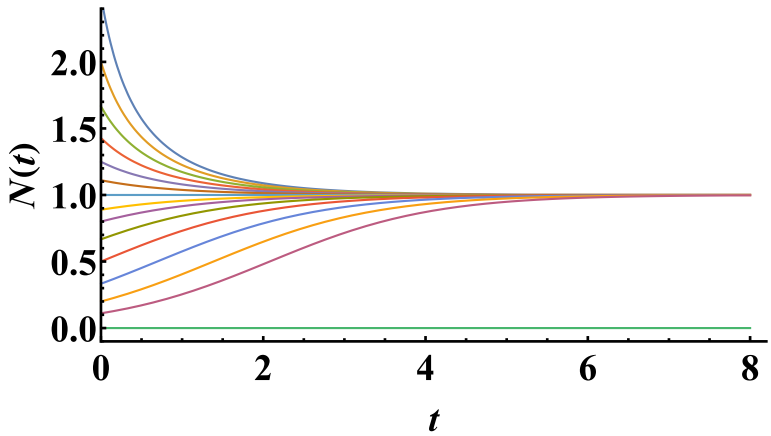


Abbildung 1: Integral curve of equation (1) for $m = 1$.

Hieraus kann gefolgert werden, dass der Prozess zwei Gleichgewichtslagen hat, $N = 0$ und $N = 1$. Zwischen den Punkten 0 und 1 ist das Feld von 0 nach 1 gerichtet und für $N > 1$ zum Punkt 1. Deshalb ist die Gleichgewichtslage 0 instabil (sobald eine Population auftritt, wächst sie) und die Gleichgewichtslage 1 stabil (jede kleinere Population wächst, jede größere schrumpft). Also bewegt sich der Prozess von jedem Anfangswert $N > 0$ zu dem stabilen Gleichgewichtspunkt $N = 1$. Die größte stabile Karpfenpopulation ist somit m.

First of all, we restrict ourself to the case that $m = 1$. We can conclude from Figure 1 that the evolution process has two equilibrium positions, $N = 0$ and $N = 1$. For $N(0) > 0$, the population of crabs, $N(t)$, converges to $N = 1$ when time evolves. Therefore, we can say that the initial population $N(0) > 0$ is unstable except $N(0) = 1$. Within the range: $1 > N(0) > 0$, the population grows towards the stable equilibrium at $N = 1$; while for $N(0) > 1$, the population shrinks towards the stable equilibrium at $N = 1$.

In general, if we allow m to be an arbitrary real number, the initial population $N(0) > 0$ will all converges towards $N = m$, which is the stable equilibrium for the population of craps.

(iii) Wir betrachten nun den Fang von Karpfen mit einer konstanten Fangquote c mit $c < m^2/4$,

$$\dot{N} = (m - N)N - c.$$

Wie viele Gleichgewichtslagen gibt es? Sind sie stabil?

Für $c \leq m/4$ gibt es zwei Gleichgewichtslagen. Die untere Gleichgewichtslage ist instabil. Wenn also aus irgendwelchen Gründen die Größe der Population den Wert der unteren Gleichgewichtslage unterschreitet, so wird die gesamte Population in endlicher Zeit aussterben. Die obere Gleichgewichtslage ist stabil. Das ist die konstante Größe, gegen die die Population bei konstanter Fangquote c strebt.

There are two equilibrium positions for $c < m^2/4$, namely N_0 and N_1, and $N_1 > N_2$

If, for some reason, the size of the population, $N(t)$, falls below N_0, $N(t)$ will approach towards $N = 0$, which implies that the entire population will die out in finite time.

On the other hand, for a population $N(t) > N_0$, the population will always evolve towards a stable equilibrium at N_1. This is the constant size that the population is aiming for with a constant catch rate c.

(iv) Bestimmen und Zeichnen Sie zu den obigen Beispielen den Phasenfluss. Bestimmen sie die Phasengeschwindigkeit.

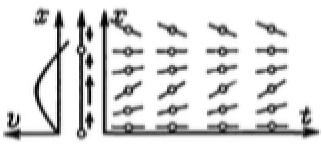


Abbildung 2: Phasenfluss und Phasengeschwindigkeit für die Populationsentwicklung ohne Fangquote für $m = 1$.

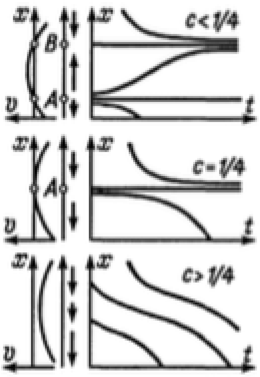


Abbildung 3: Phasenfluss und Phasengeschwindigkeit für die Populationsentwicklung mit Fangquote für $m = 1$.

Neben dem gefragten Fall, nämlich $c < m/4$ seien hier noch die beiden anderen relevanten Fangquoten, $c = m/4$ und $c > m/4$ erwähnt, für eine, bzw keine Gleichgewichtslage existieren. Für letztere Fangquote findet eine Überfischung statt. Das heißt, es wird ein größerer Anteil der Population entnommen als die maximale Vermehrungsrate, folglich stirbt die Population aus. Für den Grenzfall, dass die Fangquote genau der Vermehrungsrate entspricht gibt es eine instabile Gleichgewichtslage. Der Fang mit dieser Quote ist bei hinreichend großer Anfangspopulation mathematisch beliebig lange möglich, jedoch führt nach dem Erreichen der Gleichgewichtslage eine beliebig kleine Schwankung der Population nach unten zum vollständigen Fang der Population in endlicher Zeit.

In addition to the case in question, namely $c < m/4$, the two other relevant catch rates, $c = m/4$ and $c > m/4$, should be mentioned for one or no equilibrium situation. Overfishing is taking place for $c > m/4$. This means that the extraction rate is larger than the multiplication rate of the population. Consequently the population dies out.
There is an unstable equilibrium for $c = m/4$. This is the case when the catching rate is exactly the same as the multiplication rate of the population. Given an sufficiently large initial population, catching with this rate is mathematically possible to sustain a constant population for any length of time. However, any small fluctuation of the population can deviate the population from its equilibrium position. It the fluctuation has reduced the population, then the population will extinct within finite time.

2 Oszillatornäherung - Gleichgewichtslagen und kleine Schwingungen, Teil 1

Bevor wir in die detaillierte Diskussion linearer dynamischer Systeme einsteigen, wollen wir zunächst eine Methode diskutieren, welche die große Relevanz dieser Systeme verdeutlicht. In Teil 1 dieser Aufgabe werden wir uns dieser zunächst durch ein einfaches Beispiel nähern, bevor wir sie auf beliebige geeignete Systeme verallgemeinern. Teil 2 wird sich mit einem konkreten physikalischen Beispiel befassen.
Als Beispiel betrachten wir eine Punktmasse, die sich in folgendem Potential bewege:

$$V(x) = V_0 \cdot \left[1 - \cos\left(\frac{x}{L}\right)\right] \tag{2}$$

(i) Skizzieren Sie das Potential $V(x)$.

See Figure 4.

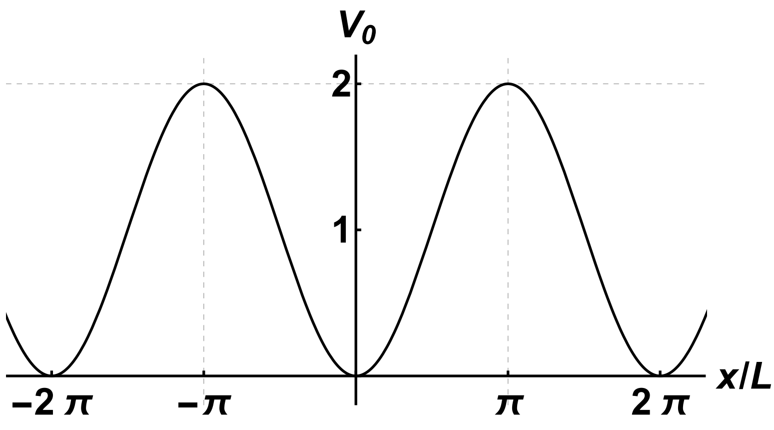


Abbildung 4: Potential profile of $V(x)$

(ii) Bestimmen Sie die Bewegungsgleichung einer Punktmasse in diesem Potential. Welche Kraft wirkt auf diese?

The force in potential $V(x)$ is given by,

$$F(x) = -\frac{d}{dx}V(x) = -\frac{V_0}{L}\sin\left(\frac{x}{L}\right)$$

so that the equation of motion is,

$$m\ddot{x}(t) = -\frac{V_0}{L}\sin\left(\frac{x}{L}\right)$$

Die *Oszillatornäherung* erlaubt die analytische Behandlung von Systemen in der Nähe ihrer *Gleichgewichtslagen*, d.h. derer Konfigurationen, in denen das System sich in Ruhe befindet. Hierfür muss inbesondere gelten, dass die Beschleunigung, die auf die einzelnen Bestandteile des Systems wirkt, verschwindet.
Hierbei muss zwischen zwei Arten von Gleichgewichtslagen unterschieden werden: *Stabile Gleichgewichtslagen* sind dadurch definiert, dass das System nach einer **beliebigen** kleinen Auslenkung wieder in die ursprüngliche Konfiguration zurückkehrt. Im Fall von *instabilen Gleichgewichtslagen* führt eine kleine Auslenkung dazu, dass das System sich weiter von der Gleichgewichtslage entfernt.

(iii) Bestimmen Sie die Gleichgewichtslagen der Punktmasse im Potential V, d.h. finden Sie die Punkte im Potential, in denen keine Kraft auf die Punktmasse wirkt. Nutzen Sie Ihre Skizze, um zu entscheiden, welche stabil bzw. instabil sind.

The equilibrium position x_0 satisfies the condition:

$$F(x_0) = 0$$

where $F(x_0) = 0$ implies that x_0 for potential $V(x)$ (Figure 4) is,

$$0 = -\frac{V_0}{L} \sin\left(\frac{x_0}{L}\right)$$

This means that x_0 is given by,

$$\frac{x_0}{L} = \begin{cases} 2n\pi, & \text{Stable equilibrium} \\ (2n+1)\pi, & \text{Unstable equilibrium} \end{cases}$$

where n is an integer.
We can determine if x_0 is stable or unstable by looking into $F(x)$. When a small perturbation δx is introduced to x_0 by $x = x_0 + \delta x$, if $F(x) \propto \delta x$, which accelerates it away from x_0, then it is an unstable equilibrium position ($V(x)$ is maximum).
On the opposite, if $F(x) \propto -\delta x$, which accelerates the particle towards x_0, then it is a stable equilibrium position ($V(x)$ is minimum). See Figure 5.

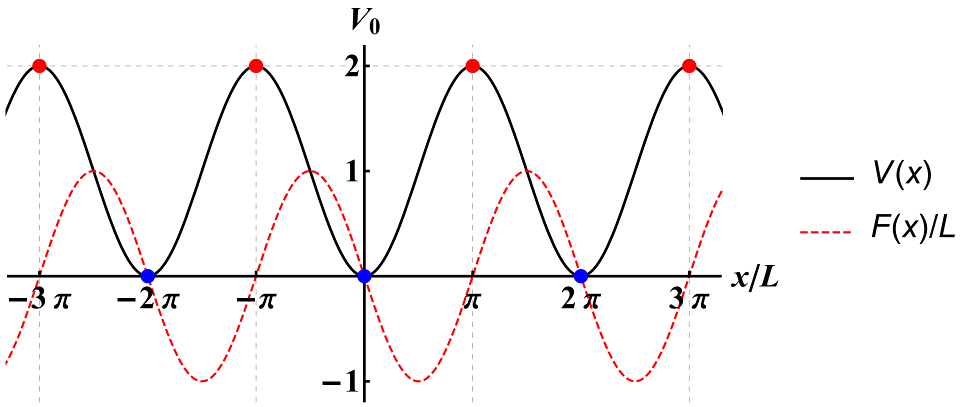


Abbildung 5: Profile of $V(x)$ and $F(x)$, red dots are unstable equilibrium while blue dots are stable equilibrium.

Wir wollen nun die Dynamik des Systems in der unmittelbaren Nähe dieser Gleichgewichtslagen betrachten. Dies erlaubt es uns, das Potential in x zu entwickeln. Wir werden sehen, dass dies in der Nähe eines Minimums zur Dynamik eines harmonischen Oszillators führt, woraus sich die Bezeichnung *Oszillatornäherung* ableitet.

(iv) Entwickeln Sie $V(x)$ bis zur zweiten Ordnung in $\frac{x}{L}$ um eine beliebige Gleichgewichtslage und nutzen Sie dieses entwickelte Potential, um die genäherten Bewegungsgleichungen abzuleiten. Zeigen Sie, dass die Entwicklung um ein Minimum zur Bewegungsgleichung des harmonischen Oszillators führt.

Expanding $V(x)$, for small x up to second order, which is a small perturbation away from the equilibrium position x_0. For stable equilibrium,

$$V(x) = \frac{V_0}{2}\left(\frac{x}{L}\right)^2$$

For unstable equilibrium,

$$\begin{aligned} V(x) &= V_0\left[1-\cos\left((2n+1)\pi+\frac{x}{L}\right)\right] = V_0\left[1+\cos\left(\frac{x}{L}\right)\right] \\ &= V_0\left[2-\frac{1}{2}\left(\frac{x}{L}\right)^2\right] \end{aligned}$$

For stable equilibrium,

$$V(x) = V_0\left[2+\frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

For stable equilibrium, the equation of motion is,

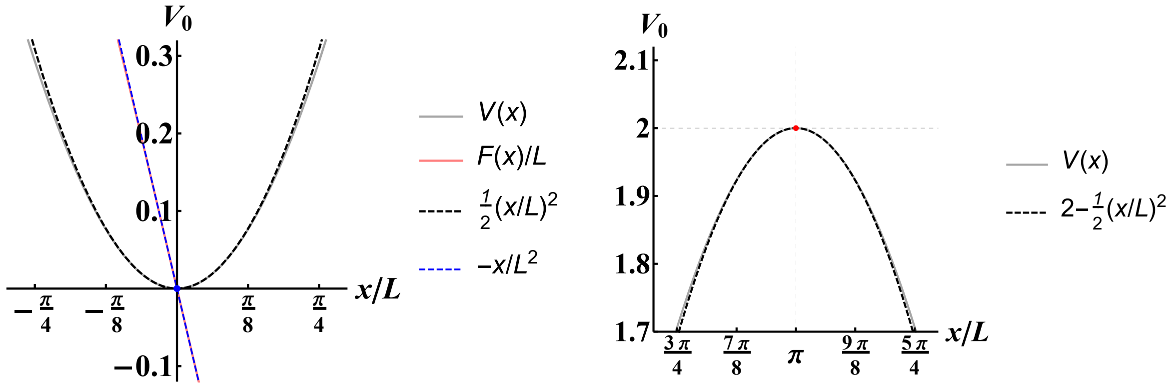
$$F(x) = -\frac{d}{dx}V(x) = -\frac{V_0}{L^2}x$$

which is similar to the equation of motion for the harmonic oscillator with $k = V_0/L^2$:

$$m\ddot{x}(t) = -kx(t)$$

(v) Ergänzen Sie Ihre Skizze aus Teilaufgabe (i) um die genäherten Potentiale um die Gleichgewichtslagen herum.

See Figure 6.



(a) $V(x)$ close to minimum.

(b) $V(x)$ close to maximum.

Abbildung 6: $V(x)$ up to second order of x for small x.

(vi) Lösen Sie die Bewegungsgleichungen unter der Annahme, dass die Punktmasse sich in der Nähe eines Minimums aufhält. Geben Sie dabei insbesondere auch die Frequenz an, mit der die Punktmasse um das Minimum oszilliert und überprüfen Sie, dass die Gleichgewichtslage im Minimum tatsächlich stabil ist.

Assuming the point mass is around the potential's minimum, for small x, the potential is expanded as,

$$V(x) = \frac{V_0}{2L^2}x^2 = \frac{1}{2}kx^2$$

where $k := V_0/L^2$, then the equation of motion is ,

$$\ddot{x}(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

The solution to the above equation of motion is,

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

where $\omega := \sqrt{k/m}$. We can see that, for t is a real number, $|x(t)| \leq \sqrt{c_1^2+c_2^2}$. Given a reasonable initial condition, for example $x(0) = 0$ and $\dot{x}(0) = \epsilon$, then we have $c_1 = 0$ and $c_2 = \omega\epsilon$. This implies that the mass oscillates around the potential minimum with an angular frequency ω, and the equilibrium position is stable.

(vii) Lösen Sie nun die Bewegungsgleichungen unter der Annahme, dass die Masse sich zunächst in der Nähe eines Maximums aufhält. Zeigen Sie explizit, dass die entsprechende Gleichgewichtslage instabil ist.

Assuming the point mass is around the potential's maximum, for small x, the potential is expanded as,

$$V(x) = V_0 \left[2 - \frac{1}{2} \left(\frac{x}{L} \right)^2 \right]$$

and the equation of motion,

$$\ddot{x}(t) = \frac{V_0}{mL^2} x(t) = \omega^2 x(t)$$

where we used the same definition that, $k := V_0/L^2$ and $\omega := \sqrt{k/m}$. The solution to the above equation of motion is,

$$x(t) = \tilde{c}_1 e^{\omega t} + \tilde{c}_2 e^{-\omega t}$$

We can see that the equilibrium position at the maximum $V(x)$ is unstable if $\tilde{c}_1 \neq 0$. In this case, $x(t)$ is unbounded because $x(t) \propto e^{\omega t}$ when $t \gg \frac{1}{\omega}$. This shows that the particle accelerates away from the equilibrium position when time evolve forward.
For a more concrete example, we look at the initial condition $x(0) = 0$ and $\dot{x}(0) = \epsilon$ where ϵ is a very tiny perturbation (push): $\epsilon \ll 1$.
Based on the initial condition, first of all we have $c_1 = -c_2$, so that,

$$x(t) = c_1 \cosh(\omega t)$$

and $c_1 = \epsilon/2\omega$ from $\dot{x}(0) = \epsilon$. This implies no matter how small the perturbation, $x(t)$ goes to infinity as $t \to \infty$.

(viii) Wir wollen nun annehmen, dass unsere Punktmasse sich zum Zeitpunkt $t = 0$ in der Nähe eines Maximums des Potentials befinde. Bestimmen Sie die Randbedingungen, für welche die Bahn der Puntkmasse die Annahme $\frac{x}{L} \ll 1$ für beliebige Zeiten $t > 0$ erfüllt. Welche Interpretation hat die entsprechende Lösung, und wieso ist sie in realen physikalischen System unwahrscheinlich?

Firstly we look at the solution,

$$x(t) = \tilde{c}_1 e^{\omega t} + \tilde{c}_2 e^{-\omega t}$$

For $t \in [0, \infty)$, the first term diverges and the second term goes to zero. For this reason, in order to preserve $x/L \ll 1$, the only possibility is to discard the first term by setting $\tilde{c}_1 = 0$.
Now we want to derive the initial condition that allows us to discard $\tilde{c}_1$. The initial condition is in general,

$$x(0) = x_0, \qquad \dot{x}(0) = v_0$$

This gives us two equations:

$$x(0) = \tilde{c}_1 + \tilde{c}_2, \qquad v_0 = \omega(\tilde{c}_1 - \tilde{c}_2)$$

Rewrites in terms of $\tilde{c}_1$,

$$\tilde{c}_1 = x_0 - \tilde{c}_2, \qquad \tilde{c}_1 = \frac{v_0}{2\omega} + \frac{x_0}{2}$$

Let us start with the trivial case where we require $\tilde{c}_1 = 0$, $x_0 = 0$ and $v_0 \neq 0$, it implies,

$$\tilde{c}_2 = 0, \qquad v_0 = -x_0\omega = 0$$

This is the case when the particle stays at the equilibrium position with zero initial velocity. Without any external perturbation, the particle is forever sitting on the equilibrium position.
On the other hand, for non-trivial case where $x_0 \neq 0$ and $v_0 \neq 0$ with $\tilde{c}_1 = 0$, it implies,

$$\tilde{c}_2 = x_0, \qquad v_0 = -x_0\omega$$

This is the scenario that the particle has a precise amount of kinetic energy to reach the potential maximum and stop exactly at the equilibrium position. The initial condition, $v_0 = -x_0\omega$, is therefore a condition that fixes the total energy of the particle. We can see that by taking square on both side and multiply by $m/2$,

$$\frac{1}{2}m(v_0)^2 = \frac{1}{2}m(-\omega x_0)^2$$

which is simply stating that the initial velocity has the exact amount of kinetic energy to overcome the potential energy for reaching the potential maximum.

Now we will answer why such configuration is unrealistic in the real physical world. Asking if it is unrealistic has two possible meaning:

a) We examine a class of such classical setup and demonstrate that they cannot work in the environment provided by the real universe.

b) We proof that achieving such configuration would contradict the laws of physics.

(a) is guaranteed by the stochastic nature of our universe, which means there are always something random to ruin your perfect setup.
(b) can be proven to contradict with the fundamental postulate of quantum mechanics, the *uncertainty principle*. Even if such beautiful setup exists alone in an empty universe, quantum fluctuation in vacuum can perturb it away from equilibrium. Therefore, achieving such setup for $t \in [0, \infty)$ requires the absence of quantum fluctuation, which violates the uncertainty principle. The same is also true classical, since thermal fluctuations will at some point always kick the particle off the top of the potential.

(ix) Verallgemeinern Sie Teilaufgaben (iii)-(vii) für ein beliebiges Potential mit Minima und/oder Maxima.

The idea is to proof that, for arbitrary potential profile, the position dependence is the same when looking at the potential profile close to minima/maxima. Consider an arbitrary potential $V(x')$, which is originated from x_0 and slightly deviated by x,

$$x' = x_0 + x$$

Then consider an additional constrain, where such arbitrary function $V(x)$ is smooth enough so that an expansion by Taylor series is allowed,

$$V(x_0 + x) = \sum_{n=0}^{\infty} \frac{d^n}{dx'^n} V(x')\bigg|_{x_0} \frac{x^n}{n!}$$

Consider only up to the second order,

$$V(x_0 + x) = V(x_0) + \frac{d}{dx'}V(x')\bigg|_{x_0} x + \frac{1}{2}\frac{d^2}{dx'^2}V(x')\bigg|_{x_0} x^2 + ...$$

For potential with minimum or maximum at x_0, it is an extremum so it satisfies the following property,

$$\frac{d}{dx'}V(x')\bigg|_{x_0} = 0$$

so that,

$$V(x_0 + x) = V(x_0) + \frac{1}{2}\frac{d^2}{dx'^2}V(x')\bigg|_{x_0} x^2 + ...$$

And the property of minimum and maximum requires that,

$$\frac{d^2}{dx'^2}V(x')\bigg|_{x_0} \neq 0, \qquad \frac{d^2}{dx'^2}V(x')\bigg|_{x_0} \begin{cases} > 0, & V(x_0) = V_{\text{min}} \\ < 0, & V(x_0) = V_{\text{max}} \end{cases}$$

Since d^2V/dx'^2 is evaluated at x_0, it becomes a constant and we can define,

$$\frac{d^2}{dx'^2}V(x')\bigg|_{x_0} = \begin{cases} k, & V(x_0) = V_{\text{min}} \\ -k, & V(x_0) = V_{\text{max}} \end{cases}$$

where k is a positive real number. Then $V(x')$ becomes simply $V(x)$,

$$V(x) = \begin{cases} V(x_0) + \frac{1}{2}kx^2, & V(x_0) = V_{\text{min}} \\ \\ V(x_0) - \frac{1}{2}kx^2, & V(x_0) = V_{\text{max}} \end{cases}$$

and the equation of motion,

$$F(x) = -\frac{d}{dx}V(x) = \begin{cases} -kx, & V(x_0) = V_{\text{min}} \\ kx, & V(x_0) = V_{\text{max}} \end{cases}$$

Up to this point, we have shown that for arbitrary function $V(x')$ which can be expanded by a Taylor series, we have construct a equation of motion as similar as those in the previous section. Thus everything else follows according to our previous works, where k is given by $V''(x_0)$.

3 Vollständigkeit

Betrachten Sie ein Punktteilchen der Masse m in folgenden Zentralkraftpotentialen:

(i) $|\boldsymbol{q}|^2$

(ii) $\frac{1}{|\boldsymbol{q}|^2}$

(iii) $\ln(|\boldsymbol{q}|)$

Überprüfen Sie diese Systeme jeweils auf Vollständigkeit analog zur Vorlesung.
In the lecture notes, the Newtonian gravitational potential is demonstrated to be an incomplete physical system. Incompleteness of a physical system from the observation that there exists an initial condition in which a point mass inevitable fall into a singularity within a finite time. In the Newtonian gravitational potential, the singularity is located at $r = 0$.
For a point mass in a potential $V(\boldsymbol{q})$ with finite amount of total energy E, the geodesic of the point particle satisfies,

$$E = \frac{1}{2}m\dot{\boldsymbol{q}}^2(t) + V(\boldsymbol{q})$$

where the total energy E is constant over time,

$$\frac{dE}{dt} = 0$$

so that the particle motion is governed by the following initial value problem:

$$\dot{\boldsymbol{q}}(t) = \pm\sqrt{\frac{2}{m}}\,[E - V(\boldsymbol{q})]^{1/2}$$

Now we analysis completeness of our physical system with specific potential profile.

(i) $V(|\boldsymbol{q}|) = |\boldsymbol{q}|^2$

The singularity created by the potential, $V(|\boldsymbol{q}|) = |\boldsymbol{q}|^2$, exists at the limit $|\boldsymbol{q}| \to \infty$,

$$\lim_{|\boldsymbol{q}|\to\infty} |\boldsymbol{q}|^2 = \infty$$

To proof that this physical system is complete, we need to show that there exists a point $|\boldsymbol{q}| = |\boldsymbol{q}|_{\text{max}}$, such that $|\dot{\boldsymbol{q}}|_{\text{max}} = 0$. This is to ensure that the point mass does not reach the singularity at $|\boldsymbol{q}| \to \infty$ in finite time. We begin with the initial value problem,

$$\dot{\boldsymbol{q}}(t) = \pm\sqrt{\frac{2}{m}}\left[E - |\boldsymbol{q}|^2(t)\right]^{1/2}$$

Since the kinetic energy can be either zero or a positive real number, $|\boldsymbol{q}|(t)$ is bounded by E:

$$E - |\boldsymbol{q}|^2(t) \geq 0, \qquad \Longrightarrow \qquad E \geq |\boldsymbol{q}|^2(t)$$

For the trivial case, $E = 0$, the only possible radial position is $|\boldsymbol{q}|(t) = 0$, which is a point mass sitting at the center of the central potential. If the $E < \infty$, there exists a turning point given by $|\dot{\boldsymbol{q}}|_{\max} = 0$,

$$0 = \sqrt{\frac{2}{m}}[E - |\boldsymbol{q}|^2(t)]^{1/2}$$

so that $|\boldsymbol{q}|_{\max} = \sqrt{E}$. This implies that the singularity at $|\boldsymbol{q}| = \infty$ cannot be reached in any finite time for finite E. For this reason, as the singularity is inaccessible, the physical system for potential $|\boldsymbol{q}|^2$ is complete.

(ii) $V(|\boldsymbol{q}|) = 1/|\boldsymbol{q}|^2$,

Similarly, the singularity exists at $|\boldsymbol{q}| = 0$ for this physical system:

$$\lim_{|\boldsymbol{q}|\to 0} \frac{1}{|\boldsymbol{q}|^2} = \infty$$

Again, to show that this physical system is complete, we need to show that there exists a point $|\boldsymbol{q}| = |\boldsymbol{q}|_{\min}$, such that $\dot{\boldsymbol{q}}_{\min} = 0$. This ensures that the point mass cannot reach the singularity at $r = 0$ in finite time. We begin with the initial value problem $\dot{\boldsymbol{q}}(t)$.

$$\dot{\boldsymbol{q}}(t) = \pm\sqrt{\frac{2}{m}}\left[E - \frac{1}{|\boldsymbol{q}|^2(t)}\right]^{1/2}$$

We require the kinetic energy of the point mass to be either zero or a positive real number, then $|\boldsymbol{q}|$ is bounded by E as follow,

$$E - \frac{1}{|\boldsymbol{q}|^2(t)} \geq 0, \qquad \Longrightarrow \qquad E \geq \frac{1}{|\boldsymbol{q}|^2(t)}$$

where we assume that the total energy E is finite, $E < \infty$. One can see that the the point mass can freely travel to $|\boldsymbol{q}|(t) \to \infty$,

$$\lim_{|\boldsymbol{q}|(t)\to\infty} \dot{\boldsymbol{q}}(t) = \pm \lim_{|\boldsymbol{q}|(t)\to\infty} \sqrt{\frac{2}{m}}\left[E - \frac{1}{|\boldsymbol{q}|^2(t)}\right]^{1/2} = \pm\sqrt{\frac{2E}{m}}$$

which is an asymptotic flat (potential) region, and the point mass travel with a finite constant velocity. On the other hand, there exists a turning point in which $\dot{\boldsymbol{q}}_{\min} = 0$,

$$0 = \sqrt{\frac{2}{m}}\left[E - \frac{1}{|\boldsymbol{q}|^2(t)}\right]^{1/2}$$

so that there exists a $|\boldsymbol{q}|_{\min} = 1/\sqrt{E}$ with $\dot{r}_{\min} = 0$ for $E < infty$. This implies that the physical system under potential $1/|\boldsymbol{q}|^2$ is complete because the point mass cannot reach the sigularity at $/|\boldsymbol{q}| = 0$ in any finite time.

(iii) $V(|\boldsymbol{q}|) = \ln(|\boldsymbol{q}|)$

Notice that the function $\ln|\boldsymbol{q}|$ is logarithmic diverging at $|\boldsymbol{q}| \to 0$ and $|\boldsymbol{q}| \to \infty$,

$$\lim_{|\boldsymbol{q}|\to 0} \ln|\boldsymbol{q}| = -\infty, \qquad \lim_{|\boldsymbol{q}|\to\infty} \ln|\boldsymbol{q}| = \infty,$$

This potential has two singularities: $|\boldsymbol{q}| \to 0$ and $|\boldsymbol{q}| \to \infty$. Therefore, to show that the physical system with such potential is complete, we need to show that there exist $r_{\min}$ and $r_{\max}$, such that $\dot{r}_{\min} = 0$ and $\dot{r}_{\max} = 0$. The physical system is incomplete if one of the condition is not satisfied.

Again we begin with the initial value problem:

$$\dot{\boldsymbol{q}}(t) = \pm\sqrt{\frac{2}{m}}\left[E - \ln|\boldsymbol{q}|(t)\right]^{1/2}$$

Similarly we assume that we have finite amount of total energy E, and a kinetic energy that is either zero of a positive real number. This implies $|\boldsymbol{q}|$ is bounded by E,

$$E - \ln|\boldsymbol{q}|(t) \geq 0, \qquad \Longrightarrow \qquad |\boldsymbol{q}|(t) \leq e^E$$

This implies that there exists a $r_{\max}$, given by $\dot{r}_{\max} = 0$,

$$0 = \sqrt{\frac{2}{m}}\left[E - \ln|\boldsymbol{q}|(t)\right]^{1/2}$$

so that,

$$|\boldsymbol{q}|_{\max} = e^E, \qquad |\dot{\boldsymbol{q}}|_{\max} = 0$$

This implies that a point mass of finite energy E cannot reach the singularity at $|\boldsymbol{q}| \to \infty$ in any finite time.

However, by setting $\dot{\boldsymbol{q}}(t) = 0$, there exists only one solution, which is $|\boldsymbol{q}|_{\max}$. To show that the physical system is complete, we require the existence of $|\boldsymbol{q}|_{\min}$ with $\dot{\boldsymbol{q}}_{\min} = 0$. From our result, there are evidences that such $|\boldsymbol{q}|_{\min}$ does not exists, and our physical system is incomplete.

a) $\dot{\boldsymbol{q}} \neq 0$ everywhere for $|\boldsymbol{q}|(t) \in [0, e^E)$.

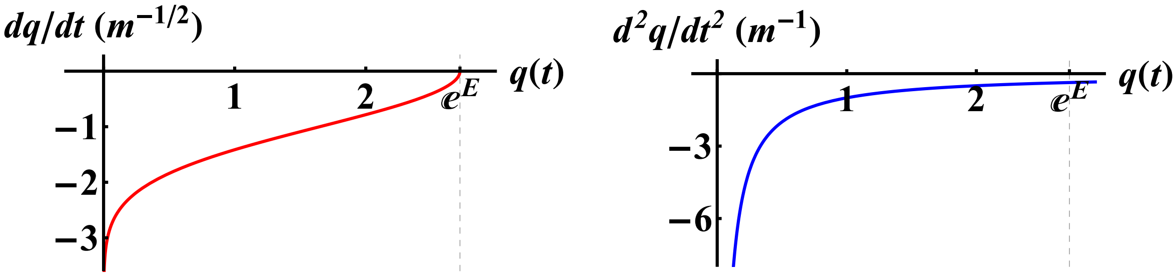
b) $\ddot{\boldsymbol{q}}(t) < 0$ for $|\boldsymbol{q}|(t) \in [0, e^E)$. For any initial condition, $\dot{\boldsymbol{q}}(t)$ accelerate continuously towards the $\boldsymbol{q} = 0$ in all $\boldsymbol{q}$. From the conservation of energy,

$$\frac{dE}{dt} = 0$$

We show that the point mass is always accelerated towards $|\boldsymbol{q}| = 0$,

$$\ddot{\boldsymbol{q}}(t) = -\frac{1}{m}\frac{d}{d\boldsymbol{q}(t)}V(|\boldsymbol{q}|) = -\frac{1}{m}\frac{1}{|\boldsymbol{q}|}$$

In the absence of angular motion, since the the point mass is always accelerating towards the center, it can reach $|\boldsymbol{q}| = 0$ with $\dot{\boldsymbol{q}} = -\infty$ in finite time, no matter what the initial conditions are.



(a) $\dot{\boldsymbol{q}}(t)$ for $\dot{\boldsymbol{q}}(t) < 0$.

(b) $\ddot{\boldsymbol{q}}(t)$ for the physical system.

Abbildung 7: $\dot{\boldsymbol{q}}(t)$ and $\ddot{\boldsymbol{q}}(t)$ for $V(|\boldsymbol{q}|) = \ln|\boldsymbol{q}|$ with $E = 1$.

This implies that there is nothing to stop the point mass from reaching the singularity at $|\boldsymbol{q}| = 0$ in finite time. One can take a further step is to integrate the initial value problem to show that the point mass can reach singularity in finite time.

3.1 Appendix

This section aims to demonstrate the idea why a point mass that reaches the singularity is defined as incomplete.

Let $t \in I := [a, b] \subset \mathbb{R}$ with $t_0 \in I$. Suppose there exists a mechanical system with an extended phase space $I \times \mathbb{P}$. The time-dependent vector field $f : I \times \mathbb{P} \to \mathbb{P}$ is *complete* if the initial value problem $f(t, x)$ posses a

unique solution for all $x \in \mathbb{P}$ and $t \in I$, for all possible initial value $x(t_0) = x_0 \in \mathbb{P}$.
This implies that f is globally Lipischitz continuous,

$$||f(t, x_1) - f(t, x_2)||_{\mathbb{P}} \leq L(t)\, ||x_1 - x_2|| \tag{3}$$

where $x_1, x_2 \in \mathbb{P}$, and L is the Lipischitz constant. For a point mass in a potential $V(\boldsymbol{q})$, the geodesic of the point particle is,

$$E = \frac{1}{2} m \dot{\boldsymbol{q}}^2(t) + V(\boldsymbol{q})$$

where E is the total finite amount of energy of the particle. This gives the initial value problem:

$$f(t, \boldsymbol{q}) = \dot{\boldsymbol{q}}(t) = \sqrt{\frac{2}{m}}\, [E - V(\boldsymbol{q})]^{1/2}$$

Notice that, if $f(t, \boldsymbol{q})$ is *differentiable*, f is Lipischitz continuous if $\partial f / \partial \boldsymbol{q}$ is bounded. In detail,

$$||f'(t, \boldsymbol{q})|| = \operatorname*{Sup}_{t \in I, \boldsymbol{q} \in \mathbb{R}^n} \left| \frac{\partial f(t, \boldsymbol{q})}{\partial \boldsymbol{q}} \right| < \infty$$

For 3 (i), (ii), (iii), the potential is a function of $|\boldsymbol{q}|$, which is not differentiable at $|\boldsymbol{q}| = 0$. Also, $\mathbb{P}$ is bounded by the total energy E, from the physical argument that the kinetic energy of ta point mass can either be zero or a positive real number:

(i) $|\boldsymbol{q}| \in [0, \sqrt{E}]$,

(ii) $|\boldsymbol{q}| \in \left[\frac{1}{\sqrt{E}}, \infty\right]$,

(iii) $|\boldsymbol{q}| \in [0, e^E]$.

To proof completeness, one needs to show:

(i) $f(t, x)$ is Lipischitz continuous everywhere except the point $|\boldsymbol{q}| = 0$ and $|\boldsymbol{q}| = |\boldsymbol{q}|_{\text{min/max}}$. This is to show $f'(t, x)$ is bounded everywhere within a rectangle $R := \{(t, x) : |t - t_0| < a, |x - x_0| < b, \}$.

(ii) $f(t, x)$ is Lipischitz continuous at $|\boldsymbol{q}| = 0$ and $|\boldsymbol{q}| = |\boldsymbol{q}|_{\text{min/max}}$ using equation (3).

One can show that for 3(iii), $\mathbb{P}$ has included a singularity which is not Lipischitz continuous for f. For this reason, the physical system is not complete because the initial value problem is not globally Lipischitz continuous for all possible initial conditions within I and $\mathbb{P}$.