Übungen zu Theoretischer Mechanik (T1)

Blatt 12

1 Hamiltonsche Systeme

Eine Punktmasse der Masse m bewege sich im dreidimensionalen Raum und sei mit einer Feder verbunden, deren zweites Ende an einem Punkt im Raum fixiert sei.

(i) Argumentieren Sie, dass die Bewegung der Punktmasse in einer Ebene stattfindet.

For a point mass connected to a spring which fixed on a point, the equation of motion is:

$$\boldsymbol{F}(t) = -k\boldsymbol{x}(t)$$

which is an equation of motion of every Cartesian component,

$$F_i(t) = -kx_i(t)$$

This harmonic motion is equivalent to a mass subjected to a central force. To show that the rotating motion is limited on a plane, consider:

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta$$

Observe that z depends only on θ. For this reason, one can orientate the Cartesian coordinate frame is a way, such tht $\cos\theta = 0$ at $\theta = \pi/2$. Then, the harmonics motion (circular motion) of the point mass is limited on the xy plane.

(ii) Überlegen Sie sich, wieso zweidimensionale Polarkoordinaten für die Beschreibung dieses Systems besonders geeignet sind.

In the absence of external force, we showed that the point mass experiences an interaction force restricted on the xy plane if we aligned the xy plane of the reference frame with the plane of rotation.

In cylindrical coordinate, the force depends only on the radius function of the point mass and the center; and secondly, the angle function describe a rotational motion under rotational symmetry. As we will see, rotational symmetry corresponds to the conservation of angular momentum. This consequently simplifies the equation of motion significantly. Accordingly, using a cylindrical coordinate has the advantage to describe a rotating motion in a simpler theoretical framework.

(iii) Bestimmen Sie die Energie des Systems in diesen Koordinaten.

$$E = T + V = \frac{m}{2}\left[\dot{r}^2(t) + r^2(t)\dot{\varphi}^2(t)\right] + \frac{1}{2}kr^2(t)$$

Die Hamiltonfunktion $H(r, p_r, \phi, p_\phi)$ dieses Systems lässt sich nun dadurch konstruieren, dass Sie die Koordinatengeschwindigkeiten in der Energie durch die dazugehörigen Impulse und ggf. auftretenden Koordinaten ersetzen, d.h. $H(r, p_r, \phi, p_\phi) = E(r, \dot{r}(p_r), \phi, \dot{\phi}(p_\phi, r))$.

(iv) Bestimmen Sie die Hamiltonfunktion dieses Systems.

Since we have

$$H(r, p_r, \phi, p_\phi) = E(r, \dot{r}(p_r), \phi, \dot{\phi}(p_\phi, r))$$

Now we replaced $\dot{\boldsymbol{q}} \leftrightarrow \boldsymbol{p}$. Recall that,

$$p_r = m\dot{r}(t), \qquad p_\varphi = mr^2\dot{\varphi}(t)$$

so that,

$$H(r, p_r, \phi, p_\phi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + \frac{1}{2}kr^2(t)$$

(v) Nutzen Sie die Hamilton-Gleichungen, um die Bewegungsgleichungen des Systems zu bestimmen.

For the Hamilton equations:

$$\frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{q}}, \qquad \frac{d\boldsymbol{q}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}}$$

where

$$\dot{p}_r(t) = \frac{p_\varphi^2}{mr^3} - kr, \qquad \dot{p}_\varphi(t) = 0, \qquad \dot{r}(t) = \frac{p_r}{m}, \qquad \dot{\varphi}(t) = \frac{p_\varphi}{mr^2}$$

The first two equation determines the equation of motion for $r(t)$ and $\varphi(t)$,

$$\ddot{r}(t) = r\dot{\varphi}^2(t) - \frac{k}{m}r \qquad \ddot{\varphi}(t) = -\frac{2\dot{\varphi}(t)\dot{r}(t)}{r}$$

(vi) Vergewissern Sie sich, dass die Hamilton-Gleichungen dieses Systems äquivalent sind zu den Newtonschen Gleichungen.

To compare with the Newtonian equation of motion, we have,

$$F_r = m\ddot{r}(t) - mr^2\dot{\varphi} + kr$$

$$F_\theta = mr\ddot{\varphi}(t) + 2m\dot{r}(t)\dot{\varphi}(t)$$

For $F_r = F_\theta = 0$, the result from Hamilton equation is equivalent to the Newtonian equation of motion.

(vii) Argumentieren Sie, dass bei der Bewegung in einem beliebigen rotationssymmetrischen Potential der Drehimpuls erhalten ist.

First of all, assume that the system has a potential with φ dependence. This implies that the Hamiltonian is:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi)$$

and the change of angular momentum,

$$\frac{dp_\varphi}{dt} = -\frac{\partial H}{\partial \varphi} = -\dot{\varphi}(t)\frac{\partial}{\partial \varphi}V(q, \varphi)$$

If a system contains a continuous rotational symmetry, it implies that the potential is invariant under infinitesimal shift of angle:

$$\varphi = \varphi_0 + \delta\varphi, \qquad V(q, \varphi_0 + \delta\varphi) = V(q, \varphi), \qquad \Longrightarrow \frac{\partial}{\partial \varphi}V(q, \varphi) = 0$$

Accordingly, if there is a continuous rotation symmetry, it indicates:

$$\frac{dp_\varphi}{dt} = -\dot{\varphi}(t)\frac{\partial}{\partial \varphi}V(q, \varphi) = 0.$$

This is being said, the angular momentum is a conserved quantity under the continuous rotational symmetry:

$$\frac{dp_\varphi}{dt} = 0, \qquad p_\varphi = \text{constant}.$$

(viii) Eliminieren Sie die Winkelvariable, um die Bewegung in Radialrichtung durch ein effektives Potential beschreiben zu können.

Eliminating the angular variables implies that $\dot{\varphi} = 0$ and thus $p_\varphi = 0$. The radial acceleration is simply,

$$\ddot{r}(t) = -\frac{k}{m} r$$

This is a harmonic motion along the radial direction:

$$r(t) = a\cos(\omega t) + b\sin(\omega t)$$

where $\omega := k/m$. Given the initial condition that at $t = 0$, $r(0) = 0$ and $\dot{r}(0) = v_r$,

$$r(t) = \frac{v_r}{\omega}\sin(\omega t)$$

2 Relativistisches Keplerproblem

Bisher wurde das Keplerproblem nicht-relativistisch betrachtet. Für die relativistische Betrachtung verwenden wir die Hamiltonfunktion eines relativistischen Teilchens der Masse m im Zentralpotential $U = -\alpha/|\boldsymbol{q}|$:

$$H(\boldsymbol{q}, \boldsymbol{p}) = \sqrt{c^2\boldsymbol{p}^2 + m^2c^4} - \frac{\alpha}{|\boldsymbol{q}|} \tag{1}$$

mit Lichtgeschwindigkeit c.

(i) Zeigen Sie, dass die Hamiltonfunktion in Polarkoordinaten geschrieben werden kann als

$$H(\boldsymbol{q}, \boldsymbol{p}) = \sqrt{c^2\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + m^2c^4} - \frac{\alpha}{r} \tag{2}$$

A straight forward way is to relate the momentum between the Cartesian and polar coordinate,

$$\boldsymbol{p}^2 = m^2(\dot{x}^2(t) + \dot{y}^2(t)) = m^2\left(\dot{r}^2(t) + r^2\dot{\varphi}^2(t)\right)$$

where the second term is replaced by the definition of angular momentum:

$$p_\varphi = mr^2\dot{\varphi}(t),$$

such that,

$$\boldsymbol{p}^2 = p_r^2 + \frac{p_\varphi}{r^2}$$

where $p_r(t) = m\dot{r}(t)$

(ii) Stellen Sie die Hamilton Gleichungen in geeigneten Koordinaten auf.

Recall that the Hamilton equation:

$$\frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{q}}, \qquad \frac{d\boldsymbol{q}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}}$$

which gives the following system of equations:

$$\dot{p}_r(t) = \frac{p_\varphi^2 c^2}{r^3}\left[\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right)c^2 + m^2c^4\right]^{-1/2} - \frac{\alpha}{r^2}, \qquad \dot{p}_\varphi(t) = 0,$$

$$\dot{r}(t) = p_r c^2\left[\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right)c^2 + m^2c^4\right]^{-1/2}$$

$$\dot{\varphi}(t) = \frac{p_\varphi c^2}{r^2}\left[\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right)c^2 + m^2c^4\right]^{-1/2}$$

(iii) Zeigen Sie analog zu bisherigen Rechnung auf vorherigen Blättern:

$$\varphi = \int dr \frac{\frac{L}{r^2}}{\sqrt{\frac{1}{c^2}(E + \frac{\alpha}{r})^2 - m^2c^2 - \frac{L^2}{r^2}}} + \text{const.} \tag{3}$$

First of all, observe that,

$$\frac{d\varphi(t)}{dr(t)} = \frac{\dot{\varphi}(t)}{\dot{r}(t)} = \frac{1}{r^2}\frac{p_\varphi}{p_r}$$

On the other hand, notice that the Hamiltonian is not explicit time dependent. This implies that the Hamiltonian describe a physical system that is constant in motion, and it equals to the total amount of energy, $H(\boldsymbol{p}, \boldsymbol{q}) = E$

$$E = \sqrt{c^2\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + m^2c^4} - \frac{\alpha}{r}$$

Rearrange the above equation, we have,

$$p_r = \sqrt{\frac{1}{c^2}\left(E + \frac{\alpha}{r}\right)^2 - m^2c^2 - \frac{p_\varphi^2}{r^2}}$$

Substitute this into $d\varphi/dr$,

$$\varphi(t) - \varphi_0 = \int_{r_0}^{r(t)} \frac{p_\varphi}{r^2}\left[\frac{1}{c^2}\left(E + \frac{\alpha}{r}\right)^2 - m^2c^2 - \frac{p_\varphi^2}{r^2}\right]^{-1/2} dr$$

where $L \equiv p_\varphi$.

(iv) Berechnen Sie das Integral, indem Sie $1/r$ substituieren. Zeigen Sie, dass gebundene Bahnen gegeben sind durch

$$r(\varphi) = \frac{c_1}{1 + c_2\cos\left(c_3(\varphi - \varphi_0)\right)}. \tag{4}$$

Welche Form haben diese Kurven? Bilden Sie den nicht-relativistischen Limes $c \to \infty$.

In order to calculate the orbital trajectory, we need to integrate over the following equation:

$$\varphi(t) - \varphi_0 = \int_{r_0}^{r(t)} \frac{p_\varphi}{r^2}\left[\frac{1}{c^2}\left(E + \frac{\alpha}{r}\right)^2 - m^2c^2 - \frac{p_\varphi^2}{r^2}\right]^{-1/2} dr$$

First of all, consider a change of variable:

$$u = \frac{p_\varphi}{r}, \qquad \frac{du}{dr} = -\frac{p_\varphi}{r^2}.$$

so that,

$$\begin{aligned}\varphi(t) - \varphi_0 &= -\int_{u_0}^{u(t)} \left[\frac{1}{c^2}\left(E + \frac{\alpha}{p_\varphi}u\right)^2 - m^2c^2 - u^2\right]^{-1/2} du \\ &:= -\int_{u_0}^{u(t)} \gamma^{-1/2}\left[\Delta - (u-a)^2\right]^{-1/2} du\end{aligned}$$

where a few computation and definitions reduce the first line into the second line:

$$\gamma := 1 - q^2, \qquad q := \frac{\alpha}{p_\varphi c}, \qquad a = \frac{qE}{c\gamma}, \qquad \Delta = a^2 + \frac{1}{\gamma}\left(\frac{E^2}{c^2} - m^2c^2\right)$$

Continue with a change of variable,

$$\frac{(u-a)}{\sqrt{\Delta}} = \cos\theta$$

We have,

$$\sqrt{\gamma}(\varphi(t)-\varphi_0) = \int_{\theta_0}^{\theta(t)} d\theta = \theta(t)$$

where the integration constant is absorbed into φ_0, and

$$\frac{1}{\sqrt{\Delta}}\left(\frac{p_\varphi}{r}+a\right) = \cos(\sqrt{\gamma}(\varphi(t)-\varphi_0))$$

so that,

$$\frac{p_\varphi}{a}\frac{1}{r} = 1 + \frac{\sqrt{\Delta}}{a}\cos(\sqrt{\gamma}(\varphi(t)-\varphi_0))$$

This is simply,

$$r = \frac{c_1}{1 + c_2\cos(c_3(\varphi-\varphi_0))}$$

for

$$c_1 = \frac{p_\varphi}{a}, \quad c_2 = \frac{\sqrt{\Delta}}{a}, \quad c_3 = \sqrt{\gamma}$$

When $c \to \infty$,

$$\gamma := 1, \qquad q = 0, \qquad a = 0, \qquad \Delta = -m^2c^2$$

Then,

$$r = \lim_{c\to\infty} \frac{p_\varphi}{\sqrt{-m^2c^2}} \frac{1}{1+\cos(\varphi-\varphi_0)} = 0.$$

One can see that this limit corresponds to infinite mass. For $r \neq 0$,

$$\begin{aligned}
\varphi(t)-\varphi_0 = \lim_{c\to\infty} &\int_{r_0}^{r(t)} \frac{p_\varphi}{r^2}\left[\frac{1}{c^2}\left(E+\frac{\alpha}{r}\right)^2 - m^2c^2 - \frac{p_\varphi^2}{r^2}\right]^{-1/2} dr \\
\approx \lim_{c\to\infty} &\int_{r_0}^{r(t)} \frac{p_\varphi}{r^2}\left[-m^2c^2 - \frac{p_\varphi^2}{r^2}\right]^{-1/2} dr \\
\approx \lim_{c\to\infty} &\int_{r_0}^{r(t)} -\frac{p_\varphi}{r^2}\frac{i}{mc} dr = 0
\end{aligned}$$

which implies there is no angular motion for $r \neq 0$.

3 Poissonklammern

Für zwei Funktionen auf dem Phasenraum f und g definiert man die Poissonklammer $\{f,g\}$ wie folgt:

$$\{f,g\} = \sum_k \left(\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k}\right). \tag{5}$$

(i) Zeigen Sie für eine Funktion $f(\boldsymbol{p},\boldsymbol{q},t)$, dass $\frac{df}{dt} = \{H,f\} + \frac{\partial f}{\partial t}$ ist.

First consider the total derivative $\frac{df}{dt}$,

$$\frac{df}{dt} = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial f}{\partial p_k}\frac{dp_k}{dt}\right) + \frac{\partial f}{\partial t}$$

and given that the time evolution is uniquely defined by the Hamilton's equations:

$$\frac{d\boldsymbol{q}}{dt} = -\frac{\partial H}{\partial \boldsymbol{q}}, \qquad \frac{d\boldsymbol{q}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}}$$

where $H := H(\boldsymbol{q}, \boldsymbol{p}, t)$ is the Hamiltonain. Then we have,

$$\begin{aligned} \frac{df}{dt} &= \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q_k} \right) + \frac{\partial f}{\partial t} \\ &= \{H, f\} + \frac{\partial f}{\partial t} \end{aligned}$$

(ii) Beweisen Sie die Jacobische Identität $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$

From the anti-symmetry property of the Poisson bracket,

$$\{f, h\} = -\{h, f\}$$

It follows,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \{f, \{g, h\}\} - \{g, \{f, h\}\} - \{\{f, g\}, h\}$$

Notice that,

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}.$$

and therefore,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

(iii) Betrachten Sie ein vollständig rotationssymmetrisches, dreidimensionales System $H = \frac{\boldsymbol{p}^2}{2m} + V(r)$. Beweisen Sie mit Hilfe der Poissonschen Klammern, dass die Komponenten des Drehimpulses L_x, L_y, L_z erhalten sind.

First of all, the component of angular momentum, L_x, L_y, L_y is defined by:

$$L_k := \epsilon_{ijk}\, q_i p_j.$$

Before we write the Poisson bracket of the angular momentum and the Hamilton function, notice the following properties:

a) $\{\boldsymbol{p}^2, L_k\}$

$$\{\boldsymbol{p}^2, L_k\} = \sum_m \left(\frac{\partial \boldsymbol{p}^2}{\partial p_m}\frac{\partial L_k}{\partial q_m} - \frac{\partial \boldsymbol{p}^2}{\partial q_m}\frac{\partial L_k}{\partial p_m} \right) = \sum_m 2\epsilon_{ijk}\delta_{im}p_m p_j = 2\epsilon_{ijk}p_i p_j$$

Notice that the above quantity is zero,

$$\epsilon_{ijk}p_i p_j = 0$$

To see this, first notice that levi-civita function is anti-symmetric,

$$\epsilon_{ijk} = -\epsilon_{jik}$$

However, the following object is invariant by exchange $i \leftrightarrow j$,

$$\epsilon_{ijk}p_i p_j = \epsilon_{jik}p_j p_i = -\epsilon_{ijk}p_j p_i = -\epsilon_{ijk}p_i p_j$$

Since this quantity is equal to its own negative, the only possibility is $\epsilon_{ijk}p_i p_j = 0$.

b) $\{V(\boldsymbol{q}), L_k\}$

For exactly the same compution, we have:

$$\{V(\boldsymbol{q}), L_k\} = \sum_m -\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}\epsilon_{ijk}\frac{\partial \boldsymbol{q}}{\partial q_m}\delta_{jm}q_i = -\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}\epsilon_{ijk}\frac{q_j q_i}{|\boldsymbol{q}|} = 0$$

The last term is zero for the same reason.

Now we want to compute the poison bracket of H and L_m, where $H = \boldsymbol{p}^2/2m + V(\boldsymbol{q})$,

$$\{H, L_m\} = \left\{ \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}) \ , \ L_m \right\}$$
$$= \frac{1}{2m}\{\boldsymbol{p}^2, L_m\} + \{V(\boldsymbol{q}), L_m\} = 0$$

The first and second term on the second line equals to zero since we already shown in (a) and (b).

(iv) Berechnen Sie die Poissonklammern L_i, L_j für alle i und j. Schreiben Sie ihr Ergebnis in Matrixschreibweise d.h. bestimmen Sie die Form der Matrizen M_i so, dass sich $\{L_i, L_j\}$ als Matrix der form $\sum_{i=1}^{3} L_i M_i$, schreiben lässt.

Recall that,

$$L_k = \epsilon_{ijk} q_i p_j, \qquad \frac{\partial L_k}{\partial p_m} = \epsilon_{ijk}\delta_{jm} q_i, \qquad \frac{\partial L_k}{\partial q_m} = \epsilon_{ijk}\delta_{im} p_j,$$

We are asked to compute,

$$\{L_i, L_j\} = \sum_m \left(\frac{\partial L_i}{\partial q_m}\frac{\partial L_j}{\partial p_m} - \frac{\partial L_i}{\partial p_m}\frac{\partial L_j}{\partial q_m} \right)$$

Again we do it for the first term:

$$\sum_m \frac{\partial L_i}{\partial q_m}\frac{\partial L_j}{\partial p_m} = \sum_m (\epsilon_{lni}\delta_{lm} p_n)(\epsilon_{rsj}\delta_{sm} q_r) = \epsilon_{mni}\epsilon_{rmj} q_r p_n$$

Notice the following property of the levi civita function,

$$\epsilon_{rmj} = -\epsilon_{mrj}, \qquad \epsilon_{mni}\epsilon_{mrj} = \delta_{nr}\delta_{ij} - \delta_{nj}\delta_{ir}$$

Thus we have,

$$\sum_m \frac{\partial L_i}{\partial q_m}\frac{\partial L_j}{\partial p_m} = -(\delta_{nr}\delta_{ij} - \delta_{nj}\delta_{ir}) q_r p_n$$

Exactly the same calculation for the second term,

$$\sum_m \frac{\partial L_i}{\partial p_m}\frac{\partial L_j}{\partial q_m} = \sum_m (\epsilon_{lni}\delta_{nm} q_l)(\epsilon_{rsj}\delta_{rm} p_s) = -(\delta_{ls}\delta_{ij} - \delta_{lj}\delta_{is}) q_l p_s$$

Making the changes $l = r$, $s = n$,

$$\sum_m \frac{\partial L_i}{\partial p_m}\frac{\partial L_j}{\partial q_m} = -(\delta_{rn}\delta_{ij} - \delta_{rj}\delta_{in}) q_r p_n$$

Then,

$$\{L_i, L_j\} = \delta_{nj}\delta_{ir} q_r p_n - \delta_{rj}\delta_{in} q_r p_n$$
$$= q_i p_j - q_j p_i = \epsilon_{ijk} q_i p_j = L_k$$

Based on this result, now we can look at different combinations of i and j,

$\{L_i, L_j\}$	L_1	L_2	L_3
L_1	0	L_3	$-L_2$
L_2	$-L_3$	0	L_1
L_3	L_2	$-L_1$	0

where the column L_1, L_2, L_3 represent L_i and the row represent L_j. From this we can express all different combination of poison bracket in matrix representation in general:

$$\{\boldsymbol{L}, \boldsymbol{L}\} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} = \sum_{i=1}^{3} L_i M_i$$

where $L_{ij} := \{L_i, L_j\}$ and the dimension of this problem now set to 3. The representation of the M_i is given by,

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad M_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix} \quad , \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(v) Es bezeichnen p und q den Impuls respektive Ort eines Teilchens. Betrachten Sie die Transformation $Q = \log(1 + \sqrt{q} \cos p)$ und $P = 2(1 + \sqrt{q} \cos p)\sqrt{q} \sin p$ und berechnen Sie $\{P, Q\}$.

By a straight forward calculation,

$$\{P, Q\} = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \frac{1}{\sqrt{q}\cos p}$$

(vi) Sei nun $Q = q^\alpha \cos(\beta p)$ und $P = q^\alpha \sin(\beta p)$. Für welche α und β ist $\{Q, P\} = 1$? Eine solche Transformation wir Kanonische Transformation genannt.

Again with a straight forward calculation:

$$\{Q, P\} = -\alpha\beta q^{2\alpha-1}.$$

For $\{Q, P\} = 1$,

$$-\alpha\beta q^{2\alpha-1} = 1.$$

This requires:

$$-\alpha\beta = 1, \qquad 2\alpha - 1 = 0.$$

which implies,

$$\alpha = \frac{1}{2}, \qquad \beta = -2.$$