Übungen zu Theoretischer Mechanik (T1)

Blatt 10

1 Oszillatornäherung - Gleichgewichtslagen und kleine Schwingungen, Episode II

In dieser Aufgabe wollen wir die Oszillatornäherung auf eine (mehr oder weniger) realistische Situation anwenden. Konkret begeben wir uns auf ein Raumschiff, das versucht, Messungen an einem schwarzen Loch durchzuführen[1]. Dabei bewegen wir uns zwangsläufig im effektiven Potential des schwarzen Lochs. Wie bereits auf dem letzten Blatt diskutiert ist dieses Potential gegeben durch

$$V_{\text{eff}}(r) = -G\frac{Mm}{r} + \frac{L^2}{2mr^2}. \tag{1}$$

M bezeichne hier die Masse des Zentralkörpers, m die des Raumschiffs und L dessen Drehimpuls. G ist wie üblich die Gravitationskonstante. Sie können diese Aufgabe auch dann vollständig bearbeiten, wenn Sie die Herleitung von (1) noch nicht vollständig nachvollzogen haben.

(i) Skizzieren Sie $V_{\text{eff}}(r)$.

It is possible to sketch $V_{\text{eff}}(r)$ without specifying G, M. Observe that,

$$V_{\text{eff}}(r) = GMm\left[-\frac{1}{r} + \frac{L^2}{2GMm^2}\frac{1}{r^2}\right] := GMm\left[-\frac{1}{r} + \frac{r_s}{r^2}\right]$$

It is for convenient that we define,

$$\frac{L^2}{2GMm^2} := r_s$$

It is defined as a radius variable because it has the unit of r. It indicates that the profile V_{eff} depends only on r_s. In this section, the physical meaning for r_s is the radial position of $V_{\text{eff}}(r_s) = 0$. It exists only when the freely falling body has finite angular momentum. In other words, if $L = 0$, $r_s = 0$. Thus, it represents the angular momentum, for example, if $r_s = 1/2$, then $L^2 = GMm^2$. See Figure 1.

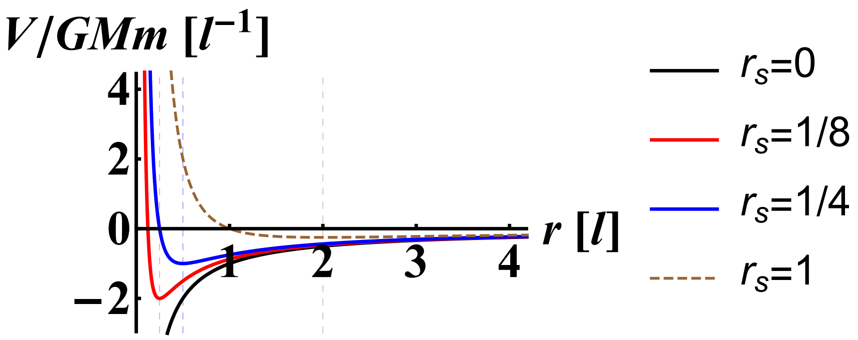


Abbildung 1: $V_{\text{eff}}(r)$ with different angular momentum

Um möglichst präzise Messungen zu gewährleisten möchten wir nun zunächst erreichen, dass unser Raumschiff sich auf einer exakten Kreisbahn mit Radius r_0 um den Zentralkörper bewegt (siehe Abb. 1).

[1] Dass hier ein schwarzes Loch verwendet wird dient einzig der Dramaturgie. Betrachtet man lediglich das Gravitationspotential macht es keinen Unterschied ob es sich beim Zentralkörper um ein schwarzes Loch oder irgendein anderes Objekt handelt.

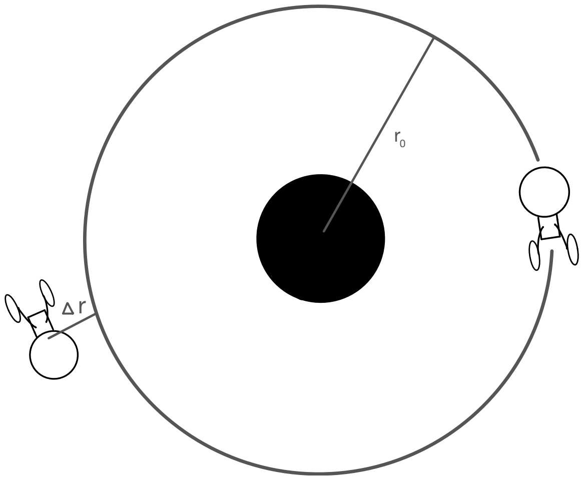


Abbildung 2: Für Teilaufgabe (ii) betrachten Sie zunächst nur die rechte Hälfte der Skizze. Die Konfiguration in der linken Bildhälfte wird erst für Teilaufgabe (iv) relevant.

(ii) Argumentieren Sie, dass der Radius r_0, für welchen das Raumschiff sich auf einer Kreisbahn bewegen kann, dem Minimum des effektiven Potentials entspricht. Geben Sie r_0 explizit an.

The extremum of V_{eff} has the following property,

$$\frac{d}{dr} V_{\text{eff}}(r) = 0$$

so there exists an extremum r_0 at,

$$r_0 = \frac{L^2}{GMm^2} = 2r_s$$

At r_0, the rotating body is at an equilibrium position,

$$F_{\text{eff}}(r) = -\frac{d}{dr} V_{\text{eff}}(r) = 0$$

Given that $V_{\text{eff}}(r_0)$ is a minimum and the rotating body is bounded by $V_{\text{eff}}(r)$ with $E < 0$. There exists two possible scenario: either it is travelling on a circular orbit at $r = r_0$, or it is on an oscillatory circular path, which fluctuate between $r = r_0 \pm \delta r$, where δr is the oscillation amplitude.

(iii) Skizzieren Sie die Bewegung auf der Kreisbahn im (r, v_r)- Phasenraum.

Along the circular orbit, $\dot{r}(t) = 0$ and $r = r_0$. See Figure 3.

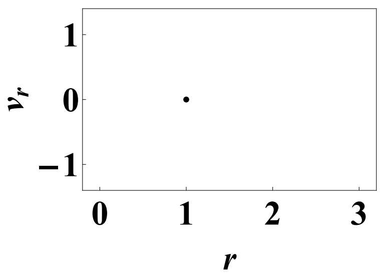


Abbildung 3: Phase space plot of a circular orbit of fixed r

Da wir nun die Kreisbahn verstanden haben wollen wir diese als Ausgangspunkt nutzen, um die Dynamik in ihrer unmittelbaren Nähe zu untersuchen. Hierzu nehmen wir an, dass unser Raumschiff um eine kleine Auslenkung Δr in seiner Bahn gestört werde, bspw. durch eine ansonsten harmlose Kollision mit einem Asteroiden (siehe Abb. 1). Um die hieraus entstehenden Störungen unserer Messungen kompensieren zu können ist es notwendig, die weitere Bahn des Schiffs vorherzusagen.

(iv) Nutzen Sie die Oszillatornäherung, um die weitere Bahn des Raumschiffs zu berechnen. Skizzieren Sie diese ebenfalls im (r, v_r)- Phasenraum.

Recall from the last exercise sheet, that, if the potential function $V_{\text{eff}}(r)$ is smooth, then it can always be expanded around its equilibrium position (extremum) r_0 by a Taylor series,

$$V_{\text{eff}}(x(t)+r_0) = V_{\text{eff}}(r_0) + \frac{d}{dr}V_{\text{eff}}(r)\bigg|_{r_0} x(t) + \frac{1}{2!}\frac{d^2}{dr^2}V_{\text{eff}}(r)\bigg|_{r_0} x^2(t) + \mathcal{O}(x^3(t))$$

where x denotes the radial perturbation around r_0, and the condition of extremum indicates that,

$$\frac{d}{dr}V_{\text{eff}}(r)\bigg|_{r_0} = 0$$

Again, if the perturbation is small enough, we can consider the potential profile near r_0 up to $\mathcal{O}(x^2(t))$. This results as an equation of motion of the perturbative motion as follow,

$$m\ddot{x}(t) = -\frac{d}{dr}V_{\text{eff}}(r) = -V''_{\text{eff}}(r_0)x(t) := -kx(t)$$

where the variables are defined by,

$$k := V''_{\text{eff}}(r_0), \qquad V''_{\text{eff}}(r_0) := \frac{d^2}{dr^2}V_{\text{eff}}(r)\bigg|_{r_0} = GMm\left[-\frac{2}{r_0^3} + \frac{6r_s}{r_0^4}\right]$$
$$= \frac{GMm}{8r_s^3}$$

Since r_0 is a minimum, it also implies,

$$V''_{\text{eff}}(r_0) > 0,$$

so that the solution of perturbative oscillation $x(t)$ around r_0 is simply,

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

where a and b are constants to be determined by the initial conditions and $\omega := \sqrt{k/m}$. For example, if $x(0) = 0$ and $\dot{(x)}(0) = \epsilon$,

$$x(t) = \frac{\epsilon}{\omega}\sin(\omega t), \qquad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{GM}{16r_s^3}}$$

For the phase space diagram, see Figure 4.

Nachdem wir die ersten Messungen erfolgreich beendet haben wollen wir versuchen, näher an das schwarze Loch heranzufliegen. Aus dem Bordcomputer erfahren wir, dass das effektive Potential in der Nähe eines schwarzen Lochs relativistische Korrekturen erfährt und gegeben ist durch

$$W_{\text{eff}}(r) = -G\frac{Mm}{r} + \frac{L^2}{2mr^2} - G\frac{L^2M}{m^2c^2r^3}, \tag{2}$$

wobei c die Lichtgeschwindigkeit bezeichne.

(v) Skizzieren Sie $W_{\text{eff}}(r)$. Dabei dürfen Sie ohne Rechnung annehmen, dass der dritte Term erst für Abstände deutlich kleiner als r_0 relevant wird und dass die potentielle Energie im Maximimum von $W_{\text{eff}}(r)$ positiv ist.

Again, we rewrite equation 2,

$$W_{\text{eff}}(r) = GMm\left[-\frac{1}{r} + \frac{L^2}{2GMm^2}\frac{1}{r^2} - \frac{L^2}{m^2c^2}\frac{1}{r^3}\right]$$
$$= GMm\left[-\frac{1}{r} + \frac{r_s}{r^2} - \frac{r_sr_g}{r^3}\right]$$

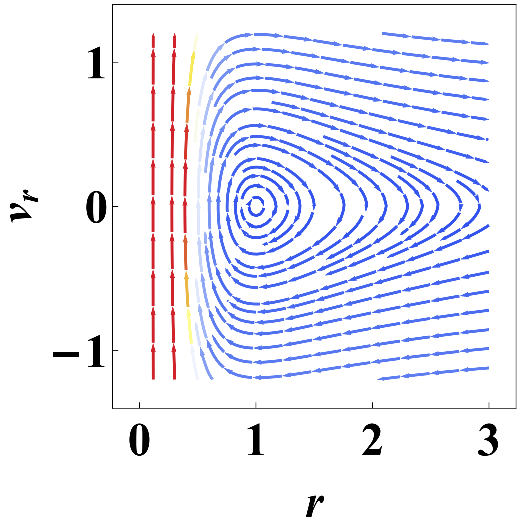
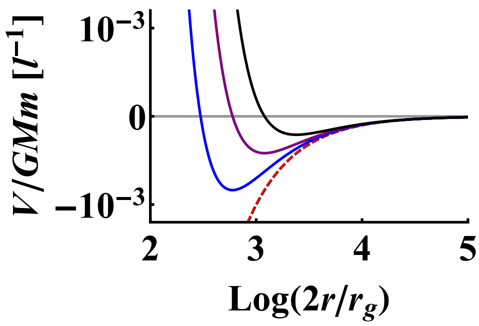
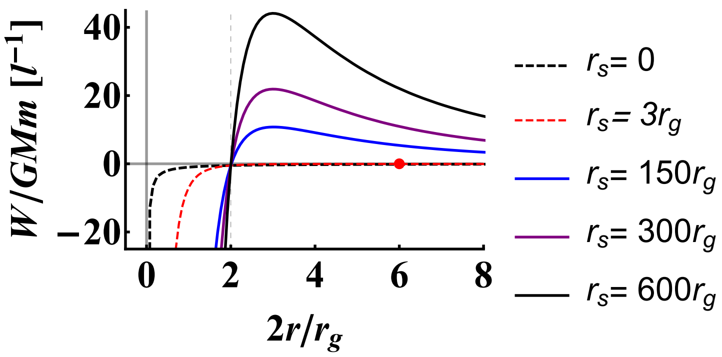


Abbildung 4: Phase space plot of a oscillating circular orbit. The color gradient represents the strength of the phase flux. It is scaled by dark blue ≈ 0 and dark red ≥ 10.



(a) $W_{\text{eff}}(r)$ for r far from r_-, showing $W_{\min}$ at r_+.



(b) $W_{\text{eff}}(r)$ for r near r_- . Red dot representing the unique stable orbital $r_{\min}$ for $L_{\min}$.

Abbildung 5: Profile of $W_{\text{eff}}(r)$.

where

$$r_s := \frac{L^2}{2GMm^2}, \qquad r_g := \frac{2GM}{c^2},$$

While r_s is characterized by angular momentum L^2; r_g is characterized by the mass of the star, M. For r_g, it is also known as the Schwarzschild radius. See Figure 5.

Wenn Sie Ihre Skizze betrachten sollten Sie feststellen, dass in diesem neuen Potential eine weitere Kreisbahn deutlich näher am schwarzen Loch realisierbar ist. Wegen der großen Vorteile von Kreisbahnen für unsere Messungen wollen wir abschließend versuchen, uns dorthin zu begeben.

(vi) Bestimmen Sie die Radien der beiden in diesem Potential möglichen Kreisbahnen. Diese werden wir als r_- und r_+ bezeichnen, wobei $r_- < r_+$.

Both circular orbit is located at the equilibrium position $r = r_\pm$ given by,

$$\frac{d}{dr} W_{\text{eff}}(r) = 0$$

so that,

$$0 = \frac{1}{r^2} - \frac{2r_s}{r^3} + \frac{3r_s r_g}{r^4}$$

This implies,

$$r_\pm = r_s \pm \sqrt{r_s\,(r_s - 3r_g)}$$

If expressed in terms of every physical quantities,

$$r_{\pm} = \frac{L^2}{2GMm^2} \pm \sqrt{\frac{L^4}{(2GMm^2)^2} - \frac{3L^2}{m^2c^2}}$$

Now, notice that the solution for $r = r_{\pm}$ exists only when $r_s \geq 3r_g$. This implies the existence of an unique stable orbit $r_{\min}$ of the 9inimum angular momentum $L = L_{\min}$,

$$r_{\min} = 3r_g, \qquad L_{\min} = \frac{\sqrt{12}GMm}{c},$$

For $L < L_{\min}$, there is no bounded orbits. See again Figure 5 for $r_{\min}$.

(vii) Nehmen Sie an, dass wir uns auf die innere Kreisbahn mit Radius r_- begeben haben. Dort werden wir erneut von einem Asteroiden getroffen, wodurch unser Raumschiff sich um δr in Richtung des schwarzen Lochs bewegt. Warum sollten wir hierüber äußerst besorgt sein?

The inner orbit, r_- corresponds to an unstable equilibrium. If a perturbation is introduced, the spaceship can either accelerates towards $r \to \infty$, or $r \to 0$. In the first case, the spaceship is not bounded by the gravity of the star and escape towards infinity. However, in the second case, the spaceship accelerates towards the singularity at $r = 0$.

(viii) Um einen Sturz ins schwarze Loch zu vermeiden entschließen wir uns, unmittelbar nach der Kollision von unseren Raumschiff aus ein Shuttle mit Masse μ in seine Richtung abzuschießen. Mit welcher Geschwindigkeit muss dies erfolgen, damit der resultierende Rückstoß unser Schiff genau zurück auf die (innere) Kreisbahn befördert?

Assuming that the spaceship gain negligible kinetic energy, δ, from the asteroid collision. When the radial position of the spaceship is shifted towards the star by δr, the amount of kinetic energy gained by this shift is,

$$T = W_{\max} - W_{\text{eff}}(r')$$

where $r' := r_- - \delta r$ and $W_{\max}$ is the maximum of the effective potential, positioned at r_-. In order to return to the unstable equilibrium position at $r = r_-$, we need a kinetic energy of $2T$: half is to decelerate the spaceship and another half is to accelerate towards r_-.

$$2\,\frac{(m\dot{r}(t))^2}{2m} = W_{\max} - W_{\text{eff}}(r')$$

Therefore, the mass μ needs to be ejected with enough momentum to provide the spaceship with energy of $2T$. Given by conservation of momentum,

$$\mu \dot{r}_\mu(t) = 2m\dot{r}(t)$$

where $\dot{r}_\mu(t)$ denotes the velocity of the ejected mass. Combining the above equations, we find $\dot{r}_\mu(t)$,

$$\dot{r}_\mu(t) = \pm\frac{2}{\mu}\sqrt{m(W_{\max} - W_{\text{eff}}(r'))}$$

and it is important that $\dot{r}_\mu(t)$ has the correct sign, which is $\dot{r}_\mu(t) < 0$.

1.1 Perturbative calculation

This section shows that the same calculation can be done in a perturbative fashion similar to exercise sheet 9.

First of all, we can expand $W_{\text{eff}}(r)$ around the equilibrium position $r = r_-$ using a Taylor series. For convenient, we define $\delta r(t) = x(t)$ temporarily,

$$W_{\text{eff}}(x(t) + r_-) = W_{\text{eff}}(r_-) + \frac{d}{dr}W_{\text{eff}}(r)\bigg|_{r_-} x(t) + \frac{1}{2!}\frac{d^2}{dr^2}W_{\text{eff}}(r)\bigg|_{r_-} x^2(t) + \mathcal{O}(x^3(t))$$

Since r_- is an extremum,

$$\frac{d}{dr}W_{\text{eff}}(r)\bigg|_{r_-} = 0$$

Following the same approach as (iv), the equation of motion for the perturbative motion around r_- is,

$$m\ddot{x}(t) = -W''_{\text{eff}}(r_-)x(t) = -kx(t)$$

where the variables are defined by,

$$k := W''_{\text{eff}}(r_-), \qquad W''_{\text{eff}}(r_-) = \frac{d^2}{dr^2}W_{\text{eff}}(r)\bigg|_{r_-} = 2GMm\left[-\frac{1}{r_-^3} + \frac{3r_s}{r_-^4} - \frac{4r_s r_g}{r_-^5}\right],$$
$$= \frac{2GMm}{r_-^5}\Big[r_s(r_g - r_-)\Big]$$

However, this time $W_{\text{eff}}(r_-)$ is a maximum, which satisfies,

$$W''_{\text{eff}}(r_-) < 0$$

Then the equation of motion becomes,

$$\ddot{x}(t) = \left|\frac{k}{m}\right| x(t)$$

where the minus sign is absorbed into k to make sure k is positive and gives $|k|$. and it is in general has a solution,

$$x(t) = ae^{\omega t} + be^{-\omega t}$$

where a, b are constants to be determined by the initial condition, and $\omega := \sqrt{|k/m|}$.

From the last exercise sheet, we showed that the solution that describes a particle with precise amount of kinetic energy to reach the potential maximum is a decaying solution. In other words, it requires $a = 0$,

$$x(t) = be^{-\omega t}$$

With the initial condition: $x(0) = x_0$, $\dot{x}(0) = v_r$, it implies,

$$b = x_0 = -\delta r, \qquad v_r = -x_0\omega = \omega\delta r$$

where $x_0 = -\delta r$ because the spaceship is perturbed r towards $r - \delta r$. So that the perturbation has a minus sign and becomes $-\delta r$. This result tells us that the initial velocity v_r needed for the spaceship to travel from $r_- - \delta r$ to r_- is,

$$v_r = -\delta r\sqrt{\left|\frac{k}{m}\right|} = -\delta r\sqrt{\left|\frac{W''_{\text{eff}}(r_-)}{m}\right|}$$

so that the ejected mass μ should be ejected with a velocity $\dot{r}_\mu(t)$ no lesser than the following,

$$\dot{r}_\mu(t) = \frac{2mv_r}{\mu} = \frac{2}{\mu}\delta r\sqrt{m\,|W''_{\text{eff}}(r_-)|}$$
$$= \frac{2}{\mu}\delta r\sqrt{\left|\frac{GMm^2}{r_-^5}\Big[r_s(r_- - r_g)\Big]\right|}$$

A factor of 2 is given because the spaceship has gained exactly the same amount of kinetic energy (or momentum) when it falls from r_- to $r_- - \delta r$. The result can be rewritten in terms of r_0,

Notice that this result is exactly the same if we expand $W_{\text{eff}}(r)$ in our previous non-perturbative calculation around r_-. Recall that the non-perturbative result is,

$$\dot{r}_\mu(t) = \frac{2}{\mu}\sqrt{W_{\text{eff}}(r_-) - W_{\text{eff}}(r')}$$

Expanding $W_{\text{eff}}(r')$ by a Talyor series and consider only up to $\mathcal{O}(\delta r^2(t))$,

$$\dot{r}_\mu(t) = \frac{2}{\mu}\sqrt{-mW''_{\text{eff}}(r_-)\delta r^2} = \frac{2}{\mu}\delta r\sqrt{m\,|W''_{\text{eff}}(r_-)|}$$

where $|W''_{\text{eff}}(r_-)|$ is given by the observation that $W_{\text{eff}}(r_-)$ is a maximum and thus $-W''_{\text{eff}}(r_-) > 0$.

(ix) Nachdem wir unsere Messungen erfolgreich beendet haben möchten wir das schwarze Loch wieder verlassen. Nutzen Sie Ihre Skizze von $W_{\text{eff}}(r)$ um zu argumentieren, dass es hierzu ausreicht, dass Raumschiff um eine beliebig kurze Strecke vom schwarzen Loch weg zu bewegen. Skizzieren Sie die sich hieraus ergebende Bewegung im Phasenraum.

Since it is an unstable equilibrium, a small perturbation towards $r > r_-$ will accelerate the spaceship away form the maximum. Such small perturbation is enough for the spaceship to escape towards $r = \infty$ because,

$$V_{\max} > V_{\text{eff}}(r), \quad \forall r \in (r_-, \infty]$$

so that $T > 0$ for all $r \in (r_-, \infty]$. Also

$$\lim_{r \to \infty} V_{\text{eff}}(r) = 0$$

so that the spaceship has a kinetic energy of $T = V_{\max}$ at $r = \infty$. The phase space diagram is in Figure 6.

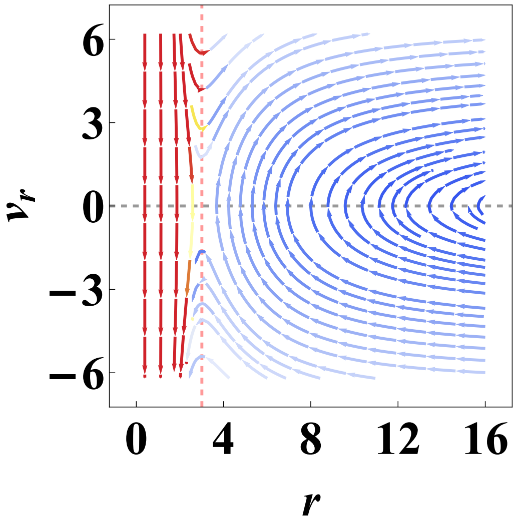


Abbildung 6: Phase space diagram near r_-. The vertical red dotted line represents $r = r_-$, which is $V_{\max}$. The color gradient represents the strength of the phase flux. It is scaled by dark blue ≈ 0 and dark red ≥ 20.

2 Zeitabhängige Frequenz

Wie Sie in der Vorlesung bereits gelernt haben, vollführt ein System welches der Bewegungsgleichung

$$\frac{d}{dt}(m\dot{x}(t)) + kx(t) = 0, \tag{3}$$

genügt harmonische Schwingungen in der Zeit. Die Parameter m und k werden, in Anlehnung an ein entsprechendes System einer gefederten Masse, oft Masse bzw. Federkonstante genannt.

(i) Bestimmen sie die Frequenz ω der Schwingungen die System (3) durchläuft.

The equation of motion is,

$$\ddot{x}(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

where ω is defined as the angular frequency,

$$\omega = \sqrt{\frac{k}{m}}$$

A easy way to see this is to observe the solution of the equation of motion,

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

where the oscillation repeats with a cycle per $\omega t = 2\pi$.

(ii) Im Allgemeinen können sowohl m als auch k von der Zeit abhängen. Argumentieren Sie weshalb man dies dennoch ohne Beschränkung der Allgemeinheit auf ein System mit konstanter Masse und Zeitabhängiger Frequenz abbilden kann.

First of all, recall from equation (3) with time-dependnet $m(t)$ and $k(t)$,

$$\frac{d}{dt}\left(m(t)\frac{d}{dt}x(t)\right) + k(t)x(t = 0$$

To map this equation of motion into a time-independent mass, we perform the following transformation (or change of variables),

$$\frac{d\tau}{dt} = \frac{1}{m(t)}$$

so that,

$$m(t)\frac{d}{dt}x(t) = \frac{d}{d\tau}x(\tau)$$

Then the equation of motion becomes,

$$m(\tau)\frac{d^2}{d\tau^2}x(\tau) + k(\tau)x(\tau) = 0\omega^2$$

This is simply a equation of motion of a harmonic oscillator with a constant mass but a time dependent angular frequency $\omega(t)$,

$$\frac{d^2}{d\tau^2}x(\tau) = -\omega^2(\tau)x(\tau)$$

where

$$\frac{k(\tau)}{m(\tau)} := \omega^2(\tau)$$

Im Weiteren gehen wir davon aus, dass die Frequenz ω eine periodische Funktion mit Periode T sei. Es ist dadurch insbesondere möglich die unabhängigen Lösungen des Problems $x_1(t), x_2(t)$ so zu wählen, dass Sie durch Funktionen der Form

$$x_1 = \alpha^{t/T} F_1(t), \quad x_2 = \beta^{t/T} F_2(t) \tag{4}$$

gegeben sind, wobei α und β Konstanten sind und die F_i periodische Funktionen der Zeit.

(iii) Zeigen Sie, dass $W(x_1, x_2) = const.$ ist, wobei W die sogenannte Wronski-Determinante ist.

First of all, recall that the Wronskian is given by,

$$W(x_1, x_2) = \frac{dx_1(\tau)}{d\tau}x_2(\tau) - \frac{dx_2(\tau)}{d\tau}x_1(\tau)$$

To show that $W(x_1, x_2)$ is a constant, it implies

$$\frac{d}{d\tau}W(x_1, x_2) = \frac{d^2x_1(\tau)}{d\tau^2}x_2(\tau) - \frac{d^2x_2(\tau)}{d\tau^2}x_1(\tau) = 0$$

This is the condition,

$$\frac{1}{x_2(\tau)}\frac{d^2}{d\tau^2}x_2(\tau) = \frac{1}{x_1(\tau)}\frac{d^2}{d\tau^2}x_1(\tau)$$

Now recall that x_1 and x_2 are the solution to the differential equation,

$$\frac{d^2}{d\tau^2}x(\tau) = -\omega(\tau)x(\tau)$$

This implies,

$$\frac{1}{x_2(\tau)}\frac{d^2}{d\tau^2}x_2(\tau) = \frac{1}{x_1(\tau)}\frac{d^2}{d\tau^2}x_1(\tau) = -\omega^2(\tau)$$

Or in other words,

$$\frac{d}{d\tau}W(x_1, x_2) = \big[-\omega(\tau) + \omega(\tau)\big]x_1(\tau)x_2(\tau) = 0$$

This shows that $W(x_1, x_2) = \text{constant}$.

(iv) Zeigen Sie, dass $\alpha\beta = 1$ und dass entweder $\alpha = \beta^*$ oder α, $\beta \in \mathbb{R}$.

Given the solution,

$$x_1(\tau) = \alpha^{\tau/T}F_1(\tau), \qquad x_2(\tau) = \beta^{\tau/T}F_2(\tau)$$

This implies that, after a period of T,

$$x_1(\tau + T) = \alpha x_1(\tau), \qquad x_2(\tau + T) = \beta x_2(\tau)$$

Since the Wronskian, $W(x_1, x_2)$ is a constant which holds for any time τ,

$$\begin{aligned} W(x_1, x_2) &= \frac{d}{d\tau}x_1(\tau + T)x_2(\tau + T) - \frac{d}{d\tau}x_2(\tau + T)x_1(\tau + T) \\ &= \alpha\beta\left[\frac{d}{d\tau}x_1(\tau)x_2(\tau) - \frac{d}{d\tau}x_2(\tau)x_1\right] = \text{constant} \end{aligned}$$

This implies the Wronskian is perseved at $\tau + T$ only when α and β satisfy the following,

$$\alpha\beta = 1$$

For the identity of α and β, notice that $x_1(t)$ and $x_2(t)$ are both the solution to the differential equation (3). If $\omega^2(\tau), \tau \in \mathbb{R}$, then the conjugate of the differential equation is,

$$\frac{d^2}{d\tau^2}x^*(\tau) = -\omega^2 x^*(\tau)$$

This implies that $x^*(t)$ also satisfies the differential equation as $x(t)$, which implies,

$$x(\tau) = x^*(\tau)$$

where

$$x(\tau) = \alpha^{\tau/T}F_1(\tau) + \beta^{\tau/T}F_2(\tau), \qquad x^*(\tau) = (\alpha^{\tau/T})^*F_1^*(\tau) + (\beta^{\tau/T})^*F_2^*(\tau)$$

where we assumed that $T \in \mathbb{R}$. If we require $x(\tau) = x^*(\tau)$, then there exists two possibilities:

a) $F_1^*(\tau) = F_1(\tau), \qquad F_2^*(\tau) = F_2(\tau), \qquad \Longrightarrow \qquad \alpha^* = \alpha, \qquad \beta^* = \beta,$

b) $F_1^*(\tau) = F_2(\tau), \qquad F_2^*(\tau) = F_1(\tau), \qquad \Longrightarrow \qquad \alpha^* = \beta, \qquad \beta^* = \alpha,$

In conclusion, from the property of the Wronskian, we showed that,

$$\alpha\beta = 1,$$

and by requiring the equation of motion to be purely real, there exists two possibilities

$$\alpha = \beta^*, \qquad \text{or}, \qquad \alpha, \beta \in \mathbb{R}$$

(v) Bestimmen Sie mittels Fallunterscheidung für obige Fälle ob das System eine stabile Gleichgewichtslage besitzt.

There are two different sceanario:

a) The first situation is $\alpha\beta = 1$ and $\alpha = \beta^*$. In this case, we have,

$$\alpha = \frac{1}{\alpha *}, \qquad \beta = \frac{1}{\beta *}$$

which indicates,

$$|\alpha|^2 = 1, \qquad |\beta|^2 = 1$$

This tells us that α and β are a complex number with a unity modulus. Then, $x_1(\tau)$ and $x_2(\tau)$ can be rewritten as:

$$x_1(\tau) = e^{i\tau\theta/T} F_1(\tau), \qquad x_2(\tau) = e^{-i\tau\theta/T} F_2(\tau)$$

where

$$\theta = \arctan \frac{\text{Im}(\alpha)}{\text{Re}(\alpha)}$$

Since $\exp[i\tau\theta/T]$ is a periodical function with unity amplitude, and $F_{1/2}(\tau)$ are a periodical function, $x_1(\tau)$ and $x_2(\tau)$ are also a periodical function. This implies that $x_(\tau)$ has a stable equilibrium position.

b) Given $\alpha\beta = 1$, and $\alpha, \beta \in \mathbb{R}$,

$$\alpha = \frac{1}{\beta}$$

Then, the independent solutions, $x_1(\tau)$ and $x_2(\tau)$ becomes,

$$x_1(\tau) = \alpha^{t/\tau} F_1(\tau), \qquad x_2(\tau) = \alpha^{-t/\tau} F_2(\tau)$$

where α is not yet determined but we know that $\alpha \in \mathbb{R}$. In this case, the general solution $x(\tau)$,

$$x(\tau) = \alpha^{t/\tau} F_1(\tau), + \alpha^{-t/\tau} F_2(\tau),$$

This indicates that for $\alpha \neq 1$, $x(\tau)$ oscillates with an amplitude that increases exponentially with time. Thus, there exists only an unstable equilibrium position, which is achievable only when $x(0) = 0$ and $\dot{x}(0) = 0$. As long as $x(0) \neq 0$ or $\dot{x} \neq 0$, the oscillations grows exponentially.

(vi) Skizzieren Sie für den Fall $\alpha, \beta \in \mathbb{R}$ die Phasenraumflüsse.

Recall that for $\alpha, \beta \in \mathbb{R}$, and $\alpha\beta = 1$, we have

$$\alpha = \frac{1}{\beta}$$

which has a general solution,

$$x(\tau) = \alpha^{\tau/T} F_1(\tau) + \alpha^{-\tau/T} F_2(\tau)$$

Assuming that the periodical function $F_1(\tau) = \cos(\Omega t)$, $F_2(\tau) = \sin(\Omega t)$, The phase space diagram parametrized by τ is in Figure 7.

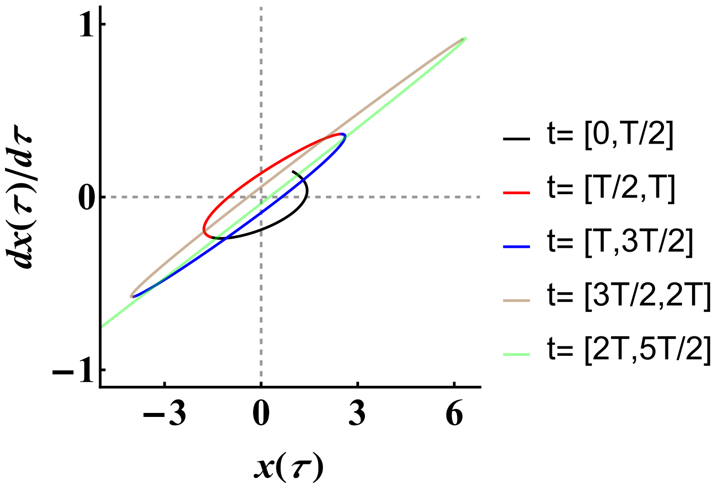


Abbildung 7: Paramteric plot of phase space for $\alpha, \beta \in \mathbb{R}$, and assuming $\alpha = 2.5$ and $\Omega = 1$.

3 Massen auf einem Ring

Betrachten Sie den unten gezeigten Aufbau dreier Massen, welche reibungsfrei auf einer kreisförmigen Schiene mit Radius r gleiten. Sie seien mit drei idealen Federn der Federkonstante k miteinander verbunden. Die Koordinaten bezeichnen hierbei die Winkel der Auslenkungen um die Ruhelagen eines Stabilen Fixpunktes.

(i) Stellen Sie die Bewegungsgleichungen des Systems in Abhängigkeit der θ_i auf. Mit den Bogenlängen $s_i = R\varphi_i$ findet man folgende Ausdrücke für die Energien,

First of all, we define:

$$s_i(t) := R\varphi_i(t), \qquad \dot{s}_i(t) = R\dot{\varphi}_i(t), \qquad s_i - s_j := \Delta s_{ij}$$

For the kinetic energy $T(\dot{\varphi}, t)$,

$$T(\dot{\varphi}, t) = \frac{1}{2}\sum_{i=1}^{3} m_i (R\dot{\varphi}_i(t))^2 = \frac{1}{2}m\Big[\dot{s}_1^2(t) + \dot{s}_2^2(t) + 3\dot{s}_3^2(t)\Big]$$

For the potential energy $U(\varphi, t)$,

$$U(\varphi, t) = \frac{1}{2}k\Big[\Delta s_{12}^2 + \Delta s_{23}^2 + \Delta s_{31}^2\Big]$$

and the total energy E of the system is given by,

$$E(\varphi, \dot{\varphi}) = T(\dot{\varphi}, t) + U(\varphi, t) = \frac{1}{2}m\Big[\dot{s}_1^2(t) + \dot{s}_2^2(t) + 3\dot{s}_3^2(t)\Big] + \frac{1}{2}k\Big[\Delta s_{12}^2 + \Delta s_{23}^2 + \Delta s_{31}^2\Big]$$

If the total energy E of the system is conserved over time, we can derive the equation of motion by,

$$\frac{dE}{dt} = 0,$$

so that,

$$0 = \Big[m\ddot{s}_1(t) + k(2s_1(t) - s_2(t) - s_3(t))\Big]\dot{s}_1(t) + \Big[m\ddot{s}_2(t) + k(2s_2(t) - s_1(t) - s_3(t))\Big]\dot{s}_2(t) + \Big[3m\ddot{s}_3(t) + k(2s_3(t) - s_1(t) - s_2(t))\Big]\dot{s}_3(t)$$

This implies that there exists three equation of motion, each corresponds the motion of each individual mass,

$$\ddot{s}_1(t) = -\frac{k}{m}(2s_1(t) - s_2(t) - s_3(t))$$
$$\ddot{s}_2(t) = -\frac{k}{m}(-s_1(t) + 2s_2(t) - s_3(t))$$
$$\ddot{s}_3(t) = -\frac{k}{3m}(-s_1(t) - s_2(t) + 2s_3(t))$$

(ii) Bestimmen Sie die Eigenfrequenzen des Systems.

In matrix representation, the equation of motion becomes,

$$\frac{d^2}{dt^2}\boldsymbol{s}(t) = \boldsymbol{K}\boldsymbol{s}(t)$$

we have,

$$\boldsymbol{s}(t) = \begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{pmatrix}, \qquad \boldsymbol{K} = \frac{k}{m}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1/3 & 1/3 & -2/3 \end{pmatrix}$$

Now we begin to analysis the eigensystem. Let $\boldsymbol{s}(t)$ be the eigenvector of the operator d^2/dt^2, which has an eigenvalue $-\omega^2$, denoting the natural frequency,

$$\frac{d^2}{dt^2}\boldsymbol{s}(t) = -\omega^2\boldsymbol{s}(t)$$

where ω can be determined by finding,

$$\det(\boldsymbol{K} + \omega^2\mathbb{I}_{3\times 3}) = 0$$

where $\mathbb{I}_{3\times 3}$ is the identity matrix, After a few straight forward calculations, solving this determinant give 5 possible eigenvalues ω,

$$\omega_1 = 0, \qquad \omega_2 = \pm\sqrt{3}\sqrt{\frac{k}{m}}, \qquad \omega_3 = \pm\sqrt{\frac{5}{3}}\sqrt{\frac{k}{m}}$$

Since ω is defined as the natural frequency, it is expected that $\omega \geq 0$, this therefore excludes $\omega < 0$.

(iii) Bestimmen Sie die Eigenschwingungen des Systems und interpretieren Sie sie.

Recall that the eigensystem is,

$$\left[\boldsymbol{K} + \omega^2\mathbb{I}_{3\times 3}\right]\boldsymbol{s}(t) = 0$$

Insert the eigenvalue ω_i into the eigensystem can solve for its corresponding eigenvector $s_i(t)$,

$$s_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad s_2(t) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad s_3(t) = \begin{pmatrix} -1 \\ -1 \\ 2/3 \end{pmatrix}$$

a) The eigenvector $s_1(t)$ belonging to the first natural frequency $\omega_1 = 0$ represents a uniform rotation of all three masses in the same direction and amplitude.

b) The eigenvector $s_2(t)$ belonging to the second natural frequency $\omega_2 = \sqrt{3k/m}$ represents an antiphase oscillation between the smaller masses with the same amplitude; while the larger mass remains at rest.

c) The eigenvector $s_3(t)$ belonging to the third natural frequency $\omega_3 = \sqrt{5k/3m}$ represents an in-phase oscillation of the smaller masses with the same amplitude; while the larger mass oscillate with antiphase compared to the smaller masses. The larger mass also oscillate with a smaller amplitude, which is a factor of $\frac{2}{3}$ compared to the smaller masses.

(iv) Geben Sie die Lösung der Bewegungsgleichungen mittels des Evolutionsoperators an.

The time evolution of the system is given by the equation of motion,

$$\frac{d^2}{dt^2}\boldsymbol{s}(t) = \boldsymbol{K}\boldsymbol{s}(t)$$

which is an eigensystem of the following,

$$\frac{d^2}{dt^2}\boldsymbol{s}(t) = -\omega^2\boldsymbol{s}(t).$$

As the system begins with $s(t_0)$, a time evolution operator $\mathcal{E}(t, t_0)$ evolves the solution forward in time from t_0 to t,

$$\boldsymbol{s}(t) = \mathcal{E}(t, t_0)\boldsymbol{s}(t_0)$$

Insert this into the equation of motion, this means,

$$\frac{d^2}{dt^2}\mathcal{E}(t, t_0) = -\omega^2 \mathcal{E}(t, t_0)$$

This implies $\mathcal{E}(t, t_0)$ with natural frequency ω is given by,

$$\mathcal{E}(t, t_0) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

so that for each eigenvector $\boldsymbol{s}_i(t)$ with natural frequency ω_i, it can be expressed in terms of the time evolution operator as follow

$$\boldsymbol{s}_i(t) = \boldsymbol{s}_i(t_0)\Big[c_1 \cos(\omega_i t) + c_2 \sin(\omega_i t)\Big]$$

where c_1 and c_2 are constants to be determined by initial conditions.