Exercises for Quantum Field Theory (TVI/TMP)

Problem Set 7

1 Axial anomaly in 1+1 dimensional QED

Consider QED in 1+1 spacetime dimensions with Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi \tag{1}$$

with covariant derivative

$$D_\mu\psi = \partial_\mu\psi - ieA_\mu\psi. \tag{2}$$

(i) Check that we can choose the Weyl representation of gamma matrices $\gamma^0 = \sigma_2$ and $\gamma^1 = i\sigma_1$. Define the chirality matrix (analogue of γ^5 in four dimensions) $\gamma_* \equiv \gamma^0\gamma^1$. Is it hermitian?

(ii) Decompose the two-component spinor as

$$\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}. \tag{3}$$

What are the equations of motion of $\psi_\pm$ in absence of electromagnetic field? Which of them is left-moving and which right-moving?

(iii) Express the vector and axial currents

$$J_V^\mu = \bar{\psi}\gamma^\mu\psi, \qquad J_A^\mu = \bar{\psi}\gamma^\mu\gamma_*\psi \tag{4}$$

in terms of $\psi_\pm$ and show that clasically they are both conserved. Which of them would still be conserved if we add a mass term for ψ?

(iv) In 1+1 space-time dimensions these two currents are not independent but are related by antisymmetric tensor $\epsilon_{\mu\nu}$ (we choose the conventions where $\epsilon_{01} = +1$). Find the relation between them.

(v) One-loop calculation of one-point function of vector current in the presence of background electromagnetic field results in

$$\int d^2x e^{iqx}\langle J_V^\mu(x)\rangle = \frac{e}{\pi}\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)A_\nu(q). \tag{5}$$

Check that this is compatible with conservation of the vector current (to which the electromagnetic field is coupled). Draw the Feynman diagram and write the expression for the corresponding amplitude. We already calculated the relevant amplitude when studying the two-point function of Dirac current. Verify the correctness of (5).

(vi) Show that instead the axial current in the background of electromagnetic field is anomalous, i.e. that

$$\langle\partial_\mu J_A^\mu\rangle_A = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{6}$$

(vii) In general, even without calculating the one-loop amplitude we expect based on Lorentz invariance the expression of the form

$$ie^2\left(A_1\eta_{\mu\nu} - A_2\frac{q_\mu q_\nu}{q^2}\right). \tag{7}$$

The coefficient A_1 is logarithmically divergent and must be regularized while the coefficient A_2 is finite, non-zero and unambiguous. The dimensional regularization results in $A_1 = A_2$. Show that this is compatible with conservation of the vector current. On the other hand, the axial current is conserved only if $A_1 = 0$ but both of these conditions cannot be satisfied at the same time if $A_2 \neq 0$.

(viii) Check that the one-loop correction to photon propagator

$$i\Pi_2^{\mu\nu}(q) = \frac{ie^2}{\pi q^2}(q^2\eta^{\mu\nu} - q^\mu q^\nu) \equiv \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) iq^2\Pi(q^2) \tag{8}$$

can be interpreted as correction to photon mass and find the corrected photon mass by considering a photon propagator with arbitrary number of insertions of electron loop (resulting in geometric series for the propagator).

2 Fermion number non-conservation

In $1+1$ dimensional electrodynamics we have the anomalous (non-)conservation law

$$\partial_\mu J_A^\mu = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{9}$$

(i) Integrate the divergence of J_A^μ and J_V^μ between two constant time slices (i.e. over all x and between times T_i and T_f) to find

$$\int d^2x\partial_\mu J_V^\mu = \Delta N_R + \Delta N_L \tag{10}$$

$$\int d^2x\partial_\mu J_A^\mu = \Delta N_R - \Delta N_L \tag{11}$$

The fact that J_V^μ is conserved will mean that the total number of left-movers plus right-movers is conserved, but the anomaly will possibly change some of the left-movers into right-movers or the other way around.

(ii) To see this at work, consider the theory on a cylinder with the space coordinate x being periodic of a finite circumference L. Let us turn on the first component of the vector potential $A^1(t,x) = A^1(t)$ such that it is independent of x and depends very slowly on t. Argue why we cannot gauge away A^1 by a periodic gauge transformation (i.e. well-defined on cylinder). You can also argue by showing that the Wilson line

$$\exp\left[ie\int_0^L A_1(t)dx\right] \tag{12}$$

is gauge invariant. On the other hand, there exist periodic gauge transformations which shift A_1 by a constant. These transformations also don't change the Wilson line. Show that actually by these transformations we can shift A_1 by $\frac{2\pi n}{eL}$ with $n \in \mathbb{N}$.

(iii) Find the Hamiltonian for ψ_+ and ψ_- in the background field A. You should find (with $A_0 = 0$)

$$H = \int dx\left[-i\psi_+^\dagger(\partial_1 - ieA_1)\psi_+ + i\psi_-^\dagger(\partial_1 - ieA_1)\psi_-\right]. \tag{13}$$

(iv) For constant A_1 the eigenfunctions of this Hamiltonian are proportional to e^{ikx} with values of k quantized (because of the periodicity in x direction). Find these allowed values of k and the corresponding eigenvalues of Hamiltonian.

(v) Now we very slowly (adiabatically) change the value of A_1. After a shift of A_1 by $\frac{2\pi}{eL}$ the spectrum of H takes the same form. If we do this change very slowly, the occupation of possible energy levels should not change (adiabaticity in quantum mechanics). Verify that as we change $A_1 \to A_1 + \frac{2\pi}{eL}$, one right-moving mode ψ_+ disappears into the vacuum while one left-moving mode ψ_- appears from the vacuum. During this process we thus have

$$\Delta N_R - \Delta N_L = -2. \tag{14}$$

(vi) Compare that result with the anomalous current conservation, i.e. evaluate

$$\int d^2x\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} \tag{15}$$

and you should find that it is exactly equal to -2.

3 Evaluation of the anomaly using point splitting

We want to calculate once again $\partial_\mu J_A^\mu$ this time using the equations of motion.

(i) At quantum level we cannot simply define the axial current as

$$J_A^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_*\psi(x) \tag{16}$$

because operators $\bar{\psi}(x)$ and $\psi(x)$ cannot be multiplied at the same point (the corresponding correlator is infinite as $x \to y$). We thus define the axial current as

$$J_A^\mu(x) = \lim_{\epsilon\to 0}\left[\bar{\psi}(x+\epsilon/2)\gamma^\mu\gamma_* \exp\left[ie\int_{x-\epsilon/2}^{x+\epsilon/2} A_\nu(z)dz^\nu\right]\psi(x-\epsilon/2)\right]. \tag{17}$$

We inserted the Wilson line not to break the gauge invariance. Verify that under gauge transformations J_A^μ defined in this way is gauge invariant.

(ii) Take the divergence of the current, use the equations of motion for ψ and $\bar{\psi}$ and find that

$$\partial_\mu J_A^\mu(x) = \lim_{\epsilon\to 0}\left[ie\bar{\psi}(x+\epsilon/2)\gamma^\mu\gamma_*\epsilon^\nu F_{\mu\nu}(x)\psi(x-\epsilon/2) + \mathcal{O}(\epsilon)\right] \tag{18}$$

(iii) Naively this would look like zero because we have a power of ϵ but don't forget that the fermion two-point function diverges at coincident points,

$$\psi(y)\bar{\psi}(z) = \frac{-i}{2\pi}\frac{\not{y}-\not{z}}{(y-z)^2}. \tag{19}$$

This implies that

$$\bar{\psi}(x+\epsilon/2)\Gamma\psi(x-\epsilon/2) = \frac{-i}{2\pi}\,\mathrm{Tr}\left(\frac{\not{\epsilon}\Gamma}{\epsilon^2}\right). \tag{20}$$

(iv) Combine the previous results together with gamma matrix identity in $1+1$ dimensions $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma_*) = -2\epsilon^{\mu\nu}$ to finally find that

$$\partial_\mu J_A^\mu = \frac{e}{2\pi}F_{\mu\nu}\epsilon^{\mu\nu}. \tag{21}$$