Prof. Ivo Sachs Summer term 2020

Exercises for Quantum Field Theory (TVI/TMP)

Problem set 3

Feynman path integral and harmonic oscillator

1 Gaussian integrals and Wick theorem

(i) Prove the Gaussian integration formula

$$\int_{\mathbb{R}^n} d^n x \exp\left[-\frac{1}{2}x^T A x + b^T x + c\right] = \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left[\frac{1}{2}b^T A^{-1} b + c\right] \tag{1}$$

valid if A is a positive definite symmetric $n \times n$ matrix and b is an arbitrary real vector.

(ii) Calculate the two-point function

$$\int_{\mathbb{R}^n} d^n x x_{j_1} x_{j_2} \exp\left[-\frac{1}{2}x^T A x\right]$$

and the four-point function

$$\int_{\mathbb{R}^n} d^n x x_{j_1} x_{j_2} x_{j_3} x_{j_4} \exp\left[-\frac{1}{2}x^T A x\right].$$

Interpret your result using Wick's theorem.

(iii) (*) Consider one-dimensional Gaussian integral (1) with $n = 1$. What are the coefficients of Taylor expansion of the RHS counting combinatorially?

2 Euclidean Propagator

The propagator (Green's function) of a massive scalar field in d dimensions satisfies

$$(-\Delta + m^2)G(x) = \delta^{(d)}(x). \tag{2}$$

Show that, by going to momentum space, that G is equal to

$$G(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{ipx}}{p^2 + m^2}. \tag{3}$$

Show that in spherical coordinates

$$G(x) = Vol(S^{d-2}) \int \frac{p^{d-1}\mathrm{d}p}{(2\pi)^d} \int_0^\pi \mathrm{d}\theta \sin^{d-2}\theta \frac{e^{ipr\cos\theta}}{p^2 + m^2} \tag{4}$$

We now specialize to $d = 3$. Perform the θ-integration and find

$$\frac{1}{(2\pi)^2 ir} \int_{-\infty}^{\infty} \mathrm{d}k k \frac{1}{k^2 + m^2} e^{ikr}. \tag{5}$$

Close the contour in the upper-half plane to pick up the pole at $+im$. You should obtain the final result

$$G(x) = \frac{1}{4\pi r} e^{-mr}. \tag{6}$$

3 Free scalar field in the canonical formalism

(i) The solution of the classical Klein-Gordon equation can be written as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ipx}), \tag{7}$$

where $p^\mu = (E_{\mathbf{p}}, \mathbf{p})$ and $p^2 = m^2$. The canonical momentum is given by $\pi = \partial_t\phi$.
We promote the fields to operators and impose canonical commutation relations

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})] = 0. \tag{8}$$

at equal time. Show that this implies that

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = 0. \tag{9}$$

(ii) Derive the Hamiltonian H by a Legendre transformation of the scalar field Lagrangian

$$L = \int d^3x(\partial_\mu\phi\partial^\mu\phi - m^2). \tag{10}$$

Express H in terms of a and $a^\dagger$. Substract an infinite constant, so that $H|0\rangle = 0$, where $|0\rangle$ is the vacuum state, i.e. $a_{\mathbf{p}}|0\rangle = 0$, for all $\mathbf{p}$.

(iii) What is the energy of the state $|\mathbf{p}_1, ..., \mathbf{p}_n\rangle = a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_n}|0\rangle$?

(iv) Verify that the (quantum) Heisenberg equations of motion are satisfied.

(v) Compute the commutator $[\phi(x^0,\mathbf{x}), \phi(y^0,\mathbf{y})]$ at non-equal times x^0 and y^0. You should find

$$[\phi(x^0,\mathbf{x}), \phi(y^0,\mathbf{y})] = D(x-y) - D(y-x), \tag{11}$$

where

$$D(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}e^{-ipx}. \tag{12}$$

(vi) Show that the measure

$$\frac{d^3\mathbf{p}}{2E_{\mathbf{p}}} \tag{13}$$

is Lorentz invariant. Hint: you can try to evaluate the integral

$$\int d^4p\Big|_{p^0>0}\delta(p^2-m^2)f(p) \tag{14}$$

with $f(p)$ any Lorentz invariant function.

(vii) Use the invariance of measure to argue that (11) is zero for space-like separations (even at non-equal times). Why doesn't this argument work for time-like separations?

(viii) Define the time-ordered two-point function

$$D_F(x-y) := \langle 0|T(\phi(x)\phi(y))|0\rangle := \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle. \tag{15}$$

Show that it can be written as

$$D_F(x) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ipx}}{p^2-m^2+i\epsilon}, \tag{16}$$

where ϵ is a small positive number, which we send to zero in the end. (*Hint:* Start from equation (16) and close the p_0 contour appropriately. Find the poles of the integrand and use Cauchy's integral theorem.)

(ix) Prove that

$$T(\phi(x)\phi(y)) = N(\phi(x)\phi(y)) + \overline{\phi(x)\phi}(y), \tag{17}$$

where

$$\overline{\phi(x)\phi}(y)) = \begin{cases} [\phi^+(x), \phi^-(y)], \ x^0 > y^0; \\ [\phi^+(y), \phi^-(x)], \ x^0 < y^0. \end{cases} \tag{18}$$

In the above expression we defined

$$\phi^+(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3\sqrt{2E_\mathbf{p}}} a_\mathbf{p} e^{-ipx}, \quad \phi^-(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3\sqrt{2E_\mathbf{p}}} a^\dagger_\mathbf{p} e^{ipx}. \tag{19}$$

The operation N is defined through its action on creation and annihilation operators. For any product of such operators, it puts annihilation operators $a_\mathbf{p}$ to the right, and creation operators $a^\dagger_\mathbf{p}$ to the left. So for example

$$N(a_\mathbf{p} a^\dagger_\mathbf{q} a_\mathbf{l}) = a^\dagger_\mathbf{q} a_\mathbf{p} a_\mathbf{l}. \tag{20}$$

You should also find that

$$\overline{\phi(x)\phi}(y) = D_F(x-y). \tag{21}$$

What is the vacuum expectation value of normal-ordered product of operators?