# Exercises for Quantum Field Theory (TVI/TMP)

## Problem set 2

## Lie algebras, Classical Yang-Mills

## 1 Lie algebras

(i) What are the Lie algebras $\mathfrak{u}(N)$, $\mathfrak{su}(N)$, $\mathfrak{o}(N)$, $\mathfrak{sl}(N,\mathbb{R})$, $\mathfrak{gl}(N,\mathbb{C})$ of Lie groups $U(N)$, $SU(N)$, $O(N)$, $SL(N,\mathbb{R})$, $GL(N,\mathbb{C})$? What are their dimensions?

**Solution:** Given a Lie group $G$, its Lie algebra $\mathfrak{g}$ is the vector space of infinitesimal transformations around the identity. More precisely, given a curve $\gamma_t \in G$, such that $\gamma_0 = 1$, we expand

$$\gamma_t = 1 - itT + \mathcal{O}(t^2). \tag{1}$$

Then the $T$ are the elements of the Lie algebra $\mathfrak{g}$. We determine the Lie algebras of the groups given in the problem.

- $U(N) := \{M \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \,|\, M^\dagger M = 1\}$. Write $M = 1 - itT + \mathcal{O}(t^2)$. Then,

$$M^\dagger M = (1 + itT^\dagger + \mathcal{O}(t^2))(1 - itT + \mathcal{O}(t^2)) = 1 + i(T^\dagger - T) + \mathcal{O}(t^2). \tag{2}$$

  The condition $M^\dagger M = 1$ forces $T = T^\dagger$. Therefore, $\mathfrak{u}(N) := \{T \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \,|\, T^\dagger = T\}$.
- $SU(N) = U(N) \cap SL(N,\mathbb{C})$, where

$$SL(N,\mathbb{C}) := \{M \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \,|\, \det M = 1\}. \tag{3}$$

  We need that $\det(1 - itT) = 1 - it\,\mathrm{tr}T + \mathcal{O}(t^2)$. We have

$$1 = \det(1 - itT + \mathcal{O}(t^2)) = 1 - it\,\mathrm{tr}T + \mathcal{O}(t^2). \tag{4}$$

  It follows that the $\mathfrak{sl}(N,\mathbb{C}) = \{T \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \,|\, \mathrm{tr}T = 0\}$. Also, $\mathfrak{su}(N) = \mathfrak{u}(N) \cap \mathfrak{sl}(N,\mathbb{C})$.
- The group $O(N)$ is like $U(N)$, but with real matrix entries. The derivation of its Lie algebra is equivalent to that of $U(N)$. It is $\mathfrak{o}(N) = \{T \in \mathrm{Mat}_{N\times N}(i\mathbb{R}) \,|\, T^\dagger = T\}$. Here, $i\mathbb{R}$ means that $T$ takes only purely imaginary values.
- $SL(N,\mathbb{R})$ is like $SL(N,\mathbb{C})$ with real matrix entries. Hence, $\mathfrak{sl}(N,\mathbb{R}) = \{T \in \mathrm{Mat}_{N\times N}(i\mathbb{R}) \,|\, \mathrm{tr}T = 0\}$.
- $GL(N,\mathbb{C}) := \{M \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \,|\, \det M \neq 0\}$, i.e. it is the group of invertible $N \times N$ matrices with complex entries. For any $T \in \mathrm{Mat}_{N\times N}(\mathbb{C})$, we can chose a small enough $t$ such that

$$\gamma_t = 1 - itT + \mathcal{O}(t^2) \tag{5}$$

  has non-zero determinant (this follows since the determinant depends continuously on its argument). Therefore, there are no restrictions put on the matrices in the Lie algebra of $GL(N,\mathbb{C})$. We conclude that $\mathfrak{gl}(N,\mathbb{C}) = \mathrm{Mat}_{N\times N}(\mathbb{C})$.

Let us discuss dimensions. Clearly, $\dim_{\mathbb{R}} \mathfrak{gl}(N,\mathbb{C}) = 2N^2$. The condition $T^\dagger = T$ makes half of the entries dependent on the other half. Therefore, $\dim_{\mathbb{R}} \mathfrak{u}(N) = N^2$. For $\mathfrak{o}(N)$, $T^\dagger = T$ forces the entries on the diagonal to be zero. The number of independent entries is therefore half the number of off-diagonal elements, i.e. $\dim_{\mathbb{R}} \mathfrak{o}(N) = \frac{N(N-1)}{2}$. The condition $\mathrm{tr}T = 0$ is a single equation. We conclude that $\dim_{\mathbb{R}} \mathfrak{sl}(N,\mathbb{C}) = 2N^2 - 2$ and $\dim_{\mathbb{R}} \mathfrak{sl}(N,\mathbb{R}) = N^2 - 1$. Finally, since the matrices in $\mathfrak{u}(N)$ have real entries, $\mathrm{tr}T = 0$ only fixes one real variable in $\mathfrak{su}(N)$. So, $\dim_{\mathbb{R}} \mathfrak{su}(N) = N^2 - 1$.

(ii) Choose a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

Consider now the 3-dimensional adjoint representation and express the generators $H$, $E$ and $F$ explicitly as $3 \times 3$ matrices. Verify that the commutation relations are the same as before.

**Solution:** We recall that a representation $(V, \rho)$ of a Lie algebra is a vector space $V$ and a map $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of Lie algebras. This means that $\rho([a,b]) = [\rho(a), \rho(b)] := \rho(a)\rho(b) - \rho(b)\rho(a)$. The adjoint representation is $V = \mathfrak{g}$ and $\rho(a)b = [a,b]$ (Check that this is indeed a Lie algebra representation by using the Jacobi identity!).

For $\mathfrak{sl}(2,\mathbb{C})$ we take (6) as a basis over the complex numbers (in general, Lie algebras do not admit a complex vector space structure, even when it is defined as a subspace of $\mathfrak{gl}(N,\mathbb{C})$. Any $\mathfrak{sl}(N,\mathbb{C})$ is an exception, since the condition $\operatorname{tr} T = 0$ is $\mathbb{C}$-linear). We compute

$$[E,F] = H, \quad [H,E] = 2E, \quad [H,F] = -2F. \tag{7}$$

These are the same equations as those satisfied by ladder operators of a harmonic oscillator. In this case, the energy difference between two nearby states would be 2. We want to find a matrix representation for $ad_H = [H,\,\cdot\,], ad_E = [E,\,\cdot\,], ad_F = [F,\,\cdot\,]$ with respect to the basis $(H,E,F) =: (e_1,e_2,e_3)$. That is, we want to determine $(ad_K)^i_j$, $K = H,E,F$ by $e_i (ad_K)^i_j = (ad_K)(e_j)$. By comparing to 6, we find

$$(ad_H) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (ad_E) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (ad_F) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}. \tag{8}$$

The commutation relatios of these matrices should still be (7). Since this is a trivial computation, we don't do it here.

**Comment:** The matrices $(ad_K)^j_k$ are actually the structure constants of the Lie algebra, up to an imaginary factor. Let $T_1 = H$, $T_2 = E$, $T_3 = F$. The structure constants are defined by

$$[T_a, T_b] = iC^c_{ab} T_c, \tag{9}$$

see also below. By comparison, we find that $(ad_{T_i})^j_k = iC^j_{ik}$.

(iii) Show that the bilinear form given by a trace in the fundamental representation

$$B_{fund}(X,Y) = \mathrm{Tr}_{fund}(XY) \tag{10}$$

is invariant in the sense that

$$B([Z,X],Y) + B(X,[Z,Y]) = 0. \tag{11}$$

for any elements $X$, $Y$ and $Z$ of the Lie algebra $\mathfrak{sl}(N,\mathbb{C})$. In the case of $\mathfrak{sl}(2,\mathbb{C})$ evaluate the components of $B_{fund}$ in the basis $E,H,F$.

**Solution:** Before we prove invariance of the trace, let us first explain why it is actually a notion of invariance. Recall that the adjoint representation of $\mathfrak{g}$ on itself is $b \mapsto -i[a,b]$. This is the infinitesimal version of the action of the Lie group $G$ on $\mathfrak{g}$, given by the following action.

$$b \mapsto Ad_{e^{-iat}}(b) = e^{-iat} b e^{iat}. \tag{12}$$

This is called the adjoint representation of the Lie *group* $G$ (the infinitesimal version is the adjoint of the Lie *algebra*). We say that a bilinear form $B(\,\cdot\,,\,\cdot\,)$ on $\mathfrak{g}$ is invariant, if it is invariant under the adjoint action of $G$, i.e.

$$(Ad_g(a), Ad_g(b)) = (a,b), \quad \forall g \in G, a, b \in \mathfrak{g}. \tag{13}$$

Writing $g = e^{-ict}$, we obtain (11) by differentiation with respect to $t$.

We prove (11) for any Lie algebra given with a matrix representation $V$ having a trace (e.g. finite dimensional). A trace satisfies

$$\mathrm{Tr}([A,B]) = 0. \tag{14}$$

Also, $[A,\,\cdot\,]$ acts as a derivation with respect to multiplication in $\mathrm{End}(V)$. Therefore,

$$0 = \mathrm{Tr}([A,BC]) = \mathrm{Tr}([A,B]C) + \mathrm{Tr}(B[A,C]). \tag{15}$$

For $\mathfrak{sl}(2,\mathbb{C})$ in the fundamental (2x2 matrix-) representation, we compute $(A,B)$ with respect to the basis (6). We find

$$(H,H) = 2, \quad (H,E) = 0, \quad (H,F) = 0, \quad (E,E) = 0, \quad (E,F) = 1, \quad (F,F) = 0. \tag{16}$$

(iv) In the case of Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ evaluate explicitly the components of the the Killing form

$$B_K(X,Y) = \mathrm{Tr}_{adj}\, \mathrm{ad}_X\, \mathrm{ad}_Y \tag{17}$$

(evaluated this time in the adjoint representation). Show that the Killing form is invariant. Here the operator $\mathrm{ad}_X$ in adjoint representation acts via Lie brackets,

$$\mathrm{ad}_X\, Y \equiv [X,Y]\,. \tag{18}$$

**Solution:** Instead of representation given in (6), we should now use (8). We find

$$\mathrm{Tr}(H,H) = 8, \quad \mathrm{Tr}(H,E) = 0, \quad \mathrm{Tr}(H,F) = 0, \quad \mathrm{Tr}(E,E) = 0, \quad \mathrm{Tr}(E,F) = 4, \quad \mathrm{Tr}(F,F) = 0. \tag{19}$$

Note that $\mathrm{Tr}_{fund}(A,B) = \frac{1}{4}\,\mathrm{Tr}_{ad}(A,B)$. For $\mathfrak{sl}(n,\mathbb{C})$, one would instead find

$$\mathrm{Tr}_{fund}(A,B) = \frac{1}{2n}\,\mathrm{Tr}_{ad}(A,B). \tag{20}$$

We show its invariance in (vii).

(v) In simple Lie algebra like $\mathfrak{sl}(2,\mathbb{C})$ all the invariant forms are proportional to each other. What is the relative normalization between the two invariant forms that we introduced in the case of $\mathfrak{sl}(2,\mathbb{C})$?

(vi) Consider a general Lie algebra with commutation relations

$$[T_a, T_b] = iC^c{}_{ab} T_c. \tag{21}$$

where $T_a$ form a basis of $\mathfrak{g}$ as a vector space. Write the matrix elements of the Killing form $B(T_a,T_b)$ in terms of the structure constants $C^c{}_{ab}$.

**Solution:** We already discussed above that, in the adjoint representation, $(T_a)^b_c = iC^b_{ac}$. Therefore,

$$\mathrm{Tr}_{ad}(T_i T_j) = T_{i}{}^{a}_{b} T_{j}{}^{b}_{a} = -C^a_{ib} C^b_{ja}. \tag{22}$$

(vii) (*) Use two times the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{23}$$

to show that the Killing form is always invariant.

**Solution:** As noted before, the Jacobi identity ensures that the adjoint representation is an actual Lie algebra representation. Let us see why.

$$[\mathrm{ad}_a, \mathrm{ad}_b](c) = \mathrm{ad}_a(\mathrm{ad}_b(c)) - \mathrm{ad}_b(\mathrm{ad}_a(c)) = [a,[b,c]] - [b,[a,c]] \overset{\text{Jacobi}}{=} [[a,b],c] = \mathrm{ad}_{[a,b]}(c). \tag{24}$$

We use this to show invariance of $B_K$.

$$\begin{aligned} B_K([a,b],c) + B_K(b,[a,c]) &= \mathrm{Tr}_{adj}(\mathrm{ad}_{[a,b]}\,\mathrm{ad}_c) + \mathrm{Tr}_{adj}(\mathrm{ad}_b\,\mathrm{ad}_{[a,c]}) & (25)\\ &= \mathrm{Tr}_{adj}([\mathrm{ad}_a,\mathrm{ad}_b]\,\mathrm{ad}_c) + \mathrm{Tr}_{adj}(\mathrm{ad}_b[\mathrm{ad}_a,\mathrm{ad}_c]) & (26)\\ &= 0. & (27)\end{aligned}$$

## 2 Gauge fields, curvature

(i) Consider a set of fields $\psi$ transforming under gauge transformations as

$$\psi(x) \mapsto \psi'(x) = U(x)\psi(x) = e^{-i\theta^a(x)T_a}\psi(x) \tag{28}$$

where $T_a$ are some matrices satisfying the commutation relations $[T_a,T_b] = iC^c_{ab}T_c$. Let us introduce a covariant derivative

$$D_\mu\psi = \partial_\mu\psi + igA^a_\mu T_a\psi \equiv \partial_\mu\psi + igA_\mu\psi. \tag{29}$$

Find the transformation law for the gauge fields $A^a_\mu$ such that $D_\mu\psi$ transforms under the gauge transformations in the same way as $\psi$.

**Comment:** This is a continuation of what was discussed on problem set 1 to the case of non-abelian gauge theories. Let $T_a$ be a set of generators of a Lie algebra $\mathfrak{g}$ in some representation. Assume we have a field $\psi(x)$ transforming in that representation. In most cases, $\psi(x)$ is itself a spinor. When we say that $\psi(x)$ transforms in some representation, we mean that each spinor component does so. So each component of the spinor $\psi(x)$ is itself a vector. On the other hand, a Lorentz transformation mixes up the spinor components, but ignores the components of related to the gauge algebra. Consider for example the theory of weak interactions. The fermion is a spinor $\psi$ transforming in the fundamental $\mathfrak{su}(2)$ representation. As such, it has eight components $\psi^{ia}$, $a = 1, 2$, $i = 1, 2, 3, 4$. The index $a$ labels the components corresponding to the gauge algebra. For example, $a = 1$ may be an electron and $a = 2$ may be a neutrino. An $SU(2)$ transformation

$$\psi^{ia} \mapsto \psi^{ia} - i\theta T^a_b \psi^{ib} + \mathcal{O}(\theta^2) \tag{30}$$

mixes the electron field and the neutrino field. On the other hand, a Lorentz transformation

$$x^\mu \mapsto \Lambda^\mu_\nu x^\nu = x^\mu - i\omega^{\rho\sigma}(M_{\rho\sigma})^\mu_\nu x^\nu + \mathcal{O}(\omega^2) \tag{31}$$

induces a transformation

$$\psi^{ia}(x) \mapsto \psi^{ia}(\Lambda^{-1}x) - \frac{i}{4}\omega^{\rho\sigma}(\Sigma_{\rho\sigma})^i_j \psi^{ja}(\Lambda^{-1}x) + \mathcal{O}(\omega^2), \tag{32}$$

where $\Sigma_{\rho\sigma} = \frac{i}{2}[\gamma_\rho, \gamma_\sigma]$. This only touches the spinor indices.

**Solution:** We make an ansatz

$$D_\mu\psi = \partial_\mu + igA_\mu\psi. \tag{33}$$

We want that this transforms as $D_\mu\psi \mapsto UD_\mu\psi$ under a gauge transformation $\psi \mapsto U\psi$. Therefore,

$$D_\mu\psi \mapsto D'_\mu U\psi \overset{!}{=} UD_\mu\psi. \tag{34}$$

Hence, we want that

$$U^{-1}D'_\mu U = D_\mu. \tag{35}$$

Explicitly,

$$U^{-1}D'_\mu U = U^{-1}(\partial_\mu + igA'_\mu)U = (\partial_\mu + U^{-1}(\partial_\mu U) + igU^{-1}A'_\mu U) \overset{!}{=} (\partial_\mu + igA_\mu). \tag{36}$$

By comparison, we find that

$$U^{-1}A'_\mu U - \frac{i}{g}U^{-1}(\partial_\mu U) = A_\mu, \tag{37}$$

or equivalently

$$A'_\mu = UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}. \tag{38}$$

Since $0 = \partial_\mu(UU^{-1}) = (\partial_\mu U)U^{-1} + U(\partial_\mu U^{-1})$, some textbooks also write

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}U(\partial_\mu U^{-1}). \tag{39}$$

(ii) Show that under infinitesimal gauge transformation we have

$$\delta A^a_\mu = \frac{1}{g}\partial_\mu\theta^a + C^a_{bc}\theta^b A^c_\mu. \tag{40}$$

**Solution:** Write $U = e^{-i\theta^a T_a}$, then

$$\frac{i}{g}(\partial_\mu U)U^{-1} = \frac{1}{g}(\partial_\mu\theta^a)T_a. \tag{41}$$

Therefore

$$\begin{aligned}
A'_\mu &= UA_\mu U^{-1} + \frac{1}{g}(\partial_\mu\theta^a)T_a = A^a_\mu UT_aU^{-1} + \frac{1}{g}(\partial_\mu\theta^a)T_a && (42)\\
&= A^a_\mu T_a - iA^a_\mu T_bT_a\theta^b + iA^a_\mu T_aT_b\theta^b + \frac{1}{g}(\partial_\mu\theta^a)T_a + \mathcal{O}(\theta^2) && (43)\\
&= A^a_\mu T_a + iA^a_\mu[T_a, T_b]\theta^b + \frac{1}{g}(\partial_\mu\theta^a)T_a + \mathcal{O}(\theta^2) && (44)\\
&= A^a_\mu T_a + i^2A^a_\mu C^c_{ab}T_c\theta^b + \frac{1}{g}(\partial_\mu\theta^a)T_a + \mathcal{O}(\theta^2) && (45)\\
&= A^a_\mu T_a - A^c_\mu C^a_{cb}T_a\theta^b + \frac{1}{g}(\partial_\mu\theta^a)T_a + \mathcal{O}(\theta^2). && (46)
\end{aligned}$$

At the infinitesimal level, we write $A'_\mu = A_\mu + \delta A_\mu = A^a_\mu T_a + \delta A^a_\mu T_a + \mathcal{O}(\theta^2)$. Therefore,

$$\delta A^a_\mu = -C^a_{cb}\theta^b A^c_\mu + \frac{1}{g}\partial_\mu\theta^a = C^a_{bc}\theta^b A^c_\mu + \frac{1}{g}\partial_\mu\theta^a. \tag{47}$$

(iii) Define the curvature tensor $G^a_{\mu\nu}$ by

$$[D_\mu, D_\nu]\,\psi = igG^a_{\mu\nu}T_a\psi \equiv igG_{\mu\nu}\psi. \tag{48}$$

Express the matrices $G_{\mu\nu}$ in terms of $A_\mu$ and the components $G^a_{\mu\nu}$ in terms of $A^a_\mu$.

**Solution:** We have $D_\mu = \partial_\mu + igA_\mu$, so

$$[D_\mu, D_\nu]\psi = [\partial_\mu + igA_\mu, \partial_\nu + igA_\nu]\psi = ig[\partial_\mu, A_\nu]\psi + ig[\partial_\nu, A_\mu]\psi - g^2[A_\mu, A_\nu]\psi \tag{49}$$

$$= ig(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]) = igG_{\mu\nu}\psi(x). \tag{50}$$

Note that the gauge field $A_\mu$ is a Lie algebra matrix. Hence, $[A_\mu, A_\nu] \neq 0$. In components we have

$$G^a_{\mu\nu}T_a = \partial_\mu A^a_\nu T_a - \partial_\nu A^a_\mu T_a + ig[T_b, T_c]A^b_\mu A^c_\nu = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gC^a_{bc}A^b_\mu A^c_\nu)T_a. \tag{51}$$

(iv) How do the quantities $G_{\mu\nu}$ transform under gauge transformations? How do $G^a_{\mu\nu}$ transform under infinitesimal gauge transformations?

**Solution:** Recall that $A'_\mu = UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$. We have

$$\partial_\mu A'_\nu = (\partial_\mu U)A_\nu U^{-1} + U(\partial_\mu A_\nu)U^{-1} + UA_\nu\partial_\mu U^{-1} + \frac{i}{g}(\partial_\mu\partial_\nu U)U^{-1} + \frac{i}{g}(\partial_\nu U)(\partial_\mu U^{-1}), \tag{52}$$

$$\partial_\nu A'_\mu = (\partial_\nu U)A_\mu U^{-1} + U(\partial_\nu A_\mu)U^{-1} + UA_\mu\partial_\nu U^{-1} + \frac{i}{g}(\partial_\nu\partial_\mu U)U^{-1} + \frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}), \tag{53}$$

$$igA'_\mu A'_\nu = igUA_\mu A_\nu U^{-1} - UA_\mu U^{-1}(\partial_\nu U)U^{-1} - (\partial_\mu U)A_\nu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}, \tag{54}$$

$$igA'_\nu A'_\mu = igUA_\nu A_\mu U^{-1} - UA_\nu U^{-1}(\partial_\mu U)U^{-1} - (\partial_\nu U)A_\mu U^{-1} - \frac{i}{g}(\partial_\nu U)U^{-1}(\partial_\mu U)U^{-1}. \tag{55}$$

$$\tag{56}$$

We can simplify these expressions using $U^{-1}(\partial_\nu U)U^{-1} = -\partial_\nu U^{-1}$. By comparison we observe that all terms involving a derivative of $U$ or $U^{-1}$ cancel. It follows that

$$G'_{\mu\nu} = UG_{\mu\nu}U^{-1}. \tag{57}$$

This means that at the infinitesimal level, $G$ transforms in the adjoint, i.e.

$$\delta G_{\mu\nu} = -ig\theta^a[T_a, G_{\mu\nu}]. \tag{58}$$

# 3 Yang-Mills action and equations of motion

(i) Show that the Lagrangian (QCD)

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\,[G_{\mu\nu}G^{\mu\nu}] + i\bar{\psi}_j(\not{D}\psi)_j - m\bar{\psi}_j\psi_j \tag{59}$$

is invariant under local gauge transformations. Here the fields $\psi$ form a vector in fundamental $N$-dimensional representation of $SU(N)$ and the invariant form is normalized such that $\mathrm{Tr}\,T_aT_b = \frac{1}{2}\delta_{ab}$.

**Solution:** Recall from the last exercise that the field strength transforms as $G \mapsto UGU^{-1}$. Therefore,

$$\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) \mapsto \mathrm{Tr}(UG_{\mu\nu}UU^{-1}G^{\mu\nu}U^{-1}) = \mathrm{Tr}(U^{-1}UG_{\mu\nu}UU^{-1}G^{\mu\nu}) = \mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}). \tag{60}$$

Also, by the properties of the covariant derivative, $D_\mu\psi \mapsto UD_\mu\psi$ under $\psi \mapsto U\psi$. Further, $\bar{\psi} \mapsto \bar{\psi}U^\dagger = \bar{\psi}U^{-1}$. From this it is obvious that

$$\bar{\psi}_j(\not{D}\psi)_j - m\bar{\psi}_j\psi_j \tag{61}$$

is also invariant.

(ii) Find the Euler-Lagrange equations of motion.

**Solution:** The action reads

$$S = \int -\frac{1}{2}\operatorname{Tr}\left[G_{\mu\nu}G^{\mu\nu}\right] + i\bar{\psi}_j(\not{D}\psi)_j - m\bar{\psi}_j\psi_j. \tag{62}$$

We first vary with respect to the conjugate spinor field $\bar{\psi}$. We find

$$\delta_{\bar{\psi}}S = \int \delta\bar{\psi}_j(i\not{D}\psi)_j - m\psi_j). \tag{63}$$

Therefore, $\psi$ has to satisfy the Dirac equation of a covariant derivative. Similarly, varying with respect to $\psi$ gives the conjugate Dirac equation, which is, as we have seen on the last problem set, equivalent to the ordinary Dirac equation. Finally, variation with respect to the gauge field gives

$$\delta_A S = \int -\operatorname{Tr}\left[G_{\mu\nu}\delta G^{\mu\nu}\right] - \delta A^a_\mu g\bar{\psi}_j T_a\gamma^\mu\psi_j. \tag{64}$$

Let us write out explicitly the variation of the field strength.

$$\delta G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[\delta A_\mu, A_\nu] + ig[A_\mu, \delta A_\nu]. \tag{65}$$

If we multiply this with $G^{\mu\nu}$, we get

$$G^{\mu\nu}\delta G_{\mu\nu} = 2G^{\mu\nu}(\partial_\mu\delta A_\nu + ig[A_\mu,\delta A_\nu]) = -2(\partial_\mu G^{\mu\nu} + ig[A_\mu, G^{\mu\nu}])\delta A_\nu + \ldots, \tag{66}$$

where "...″ involves only terms with a total derivative and a total commutator. Explicitly,

$$\ldots = 2\partial_\mu(G^{\mu\nu}\delta A_\nu) + 2ig[A_\mu, G^{\mu\nu}\delta A_\nu]. \tag{67}$$

These drop out after integration and taking the trace. Let us define

$$D_\mu G^{\mu\nu} := \partial_\mu G^{\mu\nu} + ig[A_\mu, G^{\mu\nu}]. \tag{68}$$

You can check that this defines a covariant derivative of $G$, i.e. it transforms properly under gauge transformations. After collecting all the pieces, we get

$$\delta_A S = \int 2\operatorname{Tr}(D_\mu G^{\mu\nu}\delta A_\nu) - \delta A^a_\nu g\bar{\psi}_j T_a\gamma^\nu\psi_j = \int \delta A^a_\nu((D_\mu G^{\mu\nu})_a - g\bar{\psi}_j T_a\gamma^\nu\psi_j) \tag{69}$$

We define $j^\nu_a := g\bar{\psi}_j T_a\gamma^\nu\psi_j$. Then, the equations of motions are

$$(D_\mu G^{\mu\nu})_a = j^\nu_a. \tag{70}$$

**Warning:** We may be tempted to call $j^\nu_a$ the Noether current. However, this is not the Noether current of the theory. In fact, the real Noether current has additional terms coming from terms in the field strength $G$ (these are absent in abelian gauge theories). Deriving these additional contributions is the subject of the next exercise.

(iii) Consider now the pure Yang-Mills action, i.e.

$$\mathcal{L} = -\frac{1}{4}\left[G^a_{\mu\nu}G^{\mu\nu}_a\right]. \tag{71}$$

(where due to our normalization of the gauge fields we use the metric $\delta_{ab}$ to raise and lower the indices in the adjoint representation). It is in particular invariant under the global transformations. Find the corresponding Noether currents.

**Solution:** Recall from the last exercise sheet that whenever there is a symmetry of the form

$$\mathcal{L}(\phi_j + \epsilon\delta\phi_j, \partial(\phi_j + \epsilon\delta\phi_j)) = \mathcal{L}(\phi_j, \partial\phi_j) + \epsilon\partial_\mu K^\mu(\phi_j, \partial\phi_j) + \mathcal{O}(\epsilon^2), \tag{72}$$

there is a conserved current

$$J^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j - K^\mu. \tag{73}$$

We already saw that $\left[G^a_{\mu\nu}G^{\mu\nu}_a\right]$ is invariant under local gauge transformations. Therefore, there appears no total derivative in the variation of the Lagrangian $\mathcal{L}$, i.e. $K^\mu = 0$. We only have to compute $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)}$ and $\delta_b A^a_\nu$. Note that there are actually $\dim \mathfrak{g}$ independent global gauge transformations. We write them infinitesimally as $\delta_b$.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} = -\frac{1}{2}G^{\alpha\beta}_b \frac{G^b_{\alpha\beta}}{\partial(\partial_\mu A^a_\nu)} = -\frac{1}{2}G^{\alpha\beta}_b(\delta^\mu_\alpha\delta^\nu_\beta\delta^a_b - \delta^\mu_\beta\delta^\nu_\alpha\delta^a_b) = -G^{\mu\nu}_a. \tag{74}$$

$$\delta_b A^a_\nu T_a = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha_b}\right|_{\alpha_b=0} e^{-i\alpha^c T_c} A^a_\mu T_a e^{i\alpha^d T_d} = -i[T_b, T_a]A^a_\nu = C^d_{ba}T_d A^a_\nu \;\Rightarrow\; \delta_b A^a_\nu = C^a_{bc}A^c_\nu. \tag{75}$$

The current is therefore

$$J^\mu_b = G^{\mu\nu}_a A^c_\nu C^a_{cb}. \tag{76}$$

(iv) Use the equations of motion to show that these currents are conserved.

**Solution:**

$$\partial_\mu J^\mu_b = (\partial_\mu G^{\mu\nu}_a)A^c_\nu C_{acb} + G^{\mu\nu}_a \partial_\mu A^c_\nu C_{acb} = (\partial_\mu G^{\mu\nu}_a)A^c_\nu C_{acb} + \frac{1}{2}G^{\mu\nu}_a(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)C_{acb}. \tag{77}$$

In components, the equations of motion are

$$\partial_\mu G^{\mu\nu}_a = g A^b_\mu G^{\mu\nu}_c C_{abc}. \tag{78}$$

Note that $iC_{cab} := iC^c_{ab} = \frac{1}{2}\operatorname{Tr}(T_c[T_a, T_b])$. This shows that $C_{cab}$ is anti-symmetric in all its indices. We find

$$\partial_\mu J^\mu_b = g A^d_\mu G^{\mu\nu}_e C_{ade} A^c_\nu C_{acb} + \frac{1}{2}G^{\mu\nu}_a(G^c_{\mu\nu} + gC_{cde}A^d_\mu A^e_\nu)C_{acb} \tag{79}$$

$$= \frac{g}{2}G^{\mu\nu}_e C_{eab}C_{adc}A^d_\mu A^c_\nu + \frac{g}{2}G^{\mu\nu}_e(C_{ade}C_{acb} + C_{ace}C_{adb})A^d_\mu A^c_\nu + 0 \tag{80}$$

$$= \frac{g}{2}G^{\mu\nu}_e(C_{eba}C_{acd} + (\text{cyclic in } bcd))A^d_\mu A^c_\nu \overset{\text{Jacobi}}{=} 0. \tag{81}$$