# Exercises for Quantum Field Theory (TVI/TMP)

## Problem set 1

## Dirac equation, global and local symmetries, Noether theorem

## 1 Gamma matrices

(i) Consider a four-tuple of $4 \times 4$ matrices $\gamma^\mu$, $\mu = 0, 1, 2, 3$ satisfying the Clifford algebra relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{1}_{4\times 4}. \tag{1}$$

Verify that one possible explicit representation (Weyl) of these matrices is given by

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} \\ \gamma^j &= \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \end{aligned} \tag{2}$$

where $\sigma_j$ are the usual $2 \times 2$ Pauli matrices, $\sigma_j \sigma_k = \delta_{jk} \mathbb{1}_{2\times 2} + i\epsilon_{jkl}\sigma_l$.

(ii) Argue using the defining relation (1) that the algebra of $\gamma$-matrices has a basis given by

$$\begin{aligned} &\mathbb{1} \\ &\gamma^\mu \\ \gamma^{\mu\nu} \equiv \gamma^{[\mu}\gamma^{\nu]} \equiv &\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \\ &\gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \\ &\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} \end{aligned} \tag{3}$$

How many of these matrices do we have in total? Hint: they should be as many as there are linearly independent $4 \times 4$ matrices.

(iii) Let us define the combination

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4}$$

Show that ($\epsilon^{0123} = +1$)

$$\gamma^5 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma. \tag{5}$$

Show that $\gamma^5$ anticommutes with $\gamma^\mu$ and compute $(\gamma^5)^2$.

(iv) Let us define the Dirac conjugate to be a combination of hermitian conjugation and multiplication by $\gamma^0$:

$$\bar{\psi} \equiv \psi^\dagger \gamma^0. \tag{6}$$

Verify that in our explicit representation of $\gamma$ matrices (2) we have

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu \tag{7}$$

and in particular that $\gamma^0$ is hermitian. What does it imply about $(\gamma^5)^\dagger$? Why is it impossible to find a representation of $\gamma^\mu$ where $\gamma^j$ would be hermitian?

## 2 Dirac equation

(i) The Dirac fermion is described by a 4-component vector of functions $\psi(x)$ on which the $\gamma$-matrices act by usual matrix multiplication. Show that the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{8}$$

implies that each component of $\psi$ satisfies the Klein-Gordon equation. Hint: act on the Dirac equation with $(-i\gamma^\nu \partial_\nu - m)$.

(ii) Derive the Dirac equation for conjugate spinor $\bar{\psi}(x)$.

(iii) Derive the Dirac equation using the principle of the minimal (extremal) action from Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{9}$$

## 3 Noether currents for Dirac equation

(i) (Noether's theorem) Assume that we have a Lagrangian $\mathcal{L}(\phi_j, \partial\phi_j)$ depending on fields and their first derivatives that is invariant under an infinitesimal transformation of the fields (up to a total derivative)

$$\mathcal{L}(\phi_j + \epsilon\delta\phi_j, \partial(\phi_j + \epsilon\delta\phi_j)) = \mathcal{L}(\phi_j, \partial\phi_j) + \epsilon\partial_\mu K^\mu(\phi_j, \partial\phi_j) + \mathcal{O}(\epsilon^2). \tag{10}$$

Show using the Euler-Lagrange equations that the current

$$J^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j - K^\mu \tag{11}$$

is conserved if the equations of motion are satisfied.

(ii) Notice that the Dirac action is invariant under global $U(1)$ transformations acting as

$$\psi(x) \mapsto e^{-iq\alpha}\psi(x). \tag{12}$$

where $q$ is the charge of field $\psi$. Show that the corresponding Noether current is in this case

$$J^\mu = q\bar{\psi}\gamma^\mu\psi. \tag{13}$$

Verify that it is conserved if the Dirac equation of motion is satisfied.

## 4 Local symmetries and QED

(i) Verify that the Dirac action is not invariant under local gauge transformations

$$\psi(x) \mapsto e^{-iq\alpha(x)}\psi(x) \tag{14}$$

but becomes invariant if we replace the derivative $\partial_\mu\psi$ by a covariant derivative

$$D_\mu\psi = (\partial_\mu + ieqA_\mu)\psi \tag{15}$$

where $A_\mu(x)$ is the gauge field and if we simultaneously transform the gauge field as

$$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu\alpha. \tag{16}$$

(ii) How does $D_\mu\psi$ transform under the gauge transformations?

(iii) Show that the replacement of an ordinary derivative by the covariant one is equivalent to additional coupling of the form $-eJ^\mu A_\mu$ in the Lagrangian where $J^\mu$ is the Noether current that we found previously.

(iv) Let us define the curvature (electromagnetic tensor) $F_{\mu\nu}$ such that

$$[D_\mu, D_\nu]\,\psi(x) = iqeF_{\mu\nu}(x)\psi(x). \tag{17}$$

Express $F_{\mu\nu}(x)$ in terms of $A_\mu$ and find how it transforms under the gauge transformations.

(v) The full Lagrangian of QED with Dirac matter is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi. \tag{18}$$

Find the equations of motion.