Problem Set 4:

Handout: Fri, Jun 05, 2020; Solutions: Fri, Jun 19, 2020

Problem 1 Floquet engineering

The unitary time evolution

$$\hat{U}_{\mathrm{I}}(T_i) = e^{-i\hat{\mathcal{H}}_{\mathrm{I}}T_i} \tag{1}$$

with an Ising interaction

$$\hat{\mathcal{H}}_{\mathrm{I}} = J\sum_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} \hat{S}^z_{\boldsymbol{i}}\hat{S}^z_{\boldsymbol{j}} \tag{2}$$

can be implemented in Rydberg systems, e.g. in optical tweezer arrays. Now consider the following sequence of unitary operations,

$$\hat{U}(T) = \hat{U}^\dagger_y(-\pi/2)\hat{U}^\dagger_x(\pi/2)\ \hat{U}_{\mathrm{I}}(T_3)\ \hat{U}_x(\pi/2)\ \hat{U}_{\mathrm{I}}(T_2)\ \hat{U}_y(-\pi/2)\ \hat{U}_{\mathrm{I}}(T_1), \tag{3}$$

where $\hat{U}_\mu(\phi)$ denotes a rotation in spin space around the $\mu = x, y, z$ axis by an angle ϕ.

(1.a) Describe the above sequence in words, considering spins on the Bloch sphere. Which spins interact when?

(1.b) The effective *Floquet Hamiltonian* $\hat{\mathcal{H}}_{\mathrm{eff}}$ is defined by:

$$\hat{U}(T) = e^{-i\hat{\mathcal{H}}_{\mathrm{eff}}T}, \tag{4}$$

where the total time is $T = T_1 + T_2 + T_3$.

For $T \ll 1/J$ (i.e. in the large frequency $\omega = 2\pi/T \gg J$), derive the effective Floquet Hamiltonian and show that a tunable XYZ spin model

$$\hat{\mathcal{H}}_{XYZ} = \sum_{\langle \boldsymbol{i},\boldsymbol{j}\rangle}\left[J_x\hat{S}^x_{\boldsymbol{i}}\hat{S}^x_{\boldsymbol{j}} + J_y\hat{S}^y_{\boldsymbol{i}}\hat{S}^y_{\boldsymbol{j}} + J_z\hat{S}^z_{\boldsymbol{i}}\hat{S}^z_{\boldsymbol{j}}\right] \tag{5}$$

is obtained. Derive expressions for J_x, J_y, J_z.

Problem 2 Dark states:

(2.a) Show that the Hamiltonian

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & \sqrt{n}g & 0 \\ \sqrt{n}g & \Delta & \Omega \\ 0 & \Omega & 0 \end{pmatrix} \tag{6}$$

has a so-called *dark eigenstate* with zero energy,

$$(\cos\theta, 0, -\sin\theta)^T, \qquad \tan\theta = \frac{g\sqrt{n}}{\Omega}, \tag{7}$$

for any value of Δ. Solve for the remaining eigenstates and show how they depend on Δ.

Problem 3 Electromagnetically induced transparency (EIT)

Start from the equations of motion for matter ($\hat{b}_e$, $\hat{b}_s$) and photonic fields ($\hat{\mathcal{E}}$),

$$i\partial_t \hat{b}_e = -\sqrt{n} g \hat{\mathcal{E}} + (\Delta - i\gamma_{ge})\, \hat{b}_e - \Omega \hat{b}_s \tag{8}$$

$$i\partial_t \hat{b}_s = -\Omega \hat{b}_e + (\delta - i\gamma_{gs})\, \hat{b}_s. \tag{9}$$

(3.a) Find a stationary solution with $\hat{\mathcal{P}} = \epsilon_0 \chi^{(1)} \hat{\mathcal{E}}$, where the polarization is

$$\hat{\mathcal{P}} = \sqrt{\frac{2\epsilon_0}{\omega_0}} g \sqrt{n} \hat{b}_e. \tag{10}$$

(3.b) From the solution in a), determine the linear susceptibility $\chi^{(1)}$. Find its imaginary part $\mathrm{Im}\chi^{(1)}$ (which describes the absorption of the medium) and plot it as a function of the one- and two- photon detunings Δ and δ (you may assume $\gamma_{gs} = 0$ for the plot).

Problem 4 Holstein-Primakoff representation

The algebra of spin-S operators $\hat{\boldsymbol{S}}$ can be expressed by so-called Holstein-Primakoff bosons $\hat{a}$:

$$\hat{S}^z = \hat{a}^\dagger \hat{a} - S \tag{11}$$

$$\hat{S}^+ = \hat{a}^\dagger \sqrt{2S - \hat{a}^\dagger \hat{a}} \tag{12}$$

$$\hat{S}^- = \sqrt{2S - \hat{a}^\dagger \hat{a}}\ \hat{a} \tag{13}$$

(4.a) Use the boson algebra $[\hat{a}, \hat{a}^\dagger] = 1$ to show that the spin operators defined above satisfy the spin algebra $[\hat{S}^i, \hat{S}^j] = i\epsilon_{ijk}\hat{S}^k$. It is sufficient to show: $[\hat{S}^z, \hat{S}^+] = \hat{S}^+$ and $[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$.

(4.b) Derive the Holstein-Primakoff approximation by Taylor expansion of the $\sqrt{\ }$ in the above expression. This is justified if we work with states for which $\langle \hat{a}^\dagger \hat{a} \rangle \ll S$.