Problem Set 3:

Handout: Fri, May 22, 2020; Solutions: Fri, Jun 05, 2020

Problem 1 Rydberg emission matrix elements

(1.a) The radial overlap in the hydrogen atom dipole matrix element $\langle n', \ell' | e\hat{\boldsymbol{r}} | n, \ell \rangle$ is given by:

$$\mu^r_{n'\ell',n\ell} = \int_0^\infty dr \; R_{n'\ell'}(r) r^3 R_{n\ell}(r). \tag{1}$$

Show that:

$$\mu^r_{10,n0} \simeq n^{-3/2} \tag{2}$$

using the known radial solution $R_{n\ell}(r)$.

Problem 2 Rydberg gates:

Consider a two-level atom with a long-lived ground state $|g\rangle$ and a Rydberg-excited state $|r\rangle$. In the rotating wave approximation, this system can be accurately described by the following time-dependent Hamiltonian,

$$\hat{\mathcal{H}}(t) = \left[\Omega(t)|g\rangle\langle r| + \text{h.c.}\right] + \Delta|r\rangle\langle r|, \tag{3}$$

where $\Omega(t)$ is a time-dependent Rabi frequency and Δ denotes the detuning of this Rabi drive from the $|g\rangle \to |r\rangle$ transition.

(2.a) Consider a one-qubit π-phase gate with a simplified Rabi-frequency sequence:

$$\Omega(t) = \begin{cases} \Omega_0 & 0 \leq t \leq T_\pi \\ 0 & \text{else.} \end{cases} \tag{4}$$

For $|\Psi(0)\rangle = |g\rangle$, calculate $|\Psi(T_\pi)\rangle$ as a function of the detuning Δ. Choose the value of T_π appropriately, in order to obtain a π-pulse.

(2.b) Using (a), discuss under which conditions the Rydberg blockade works and derive the condition for the blockade radius. Assume for simplicity, that the lifetime τ_r of the Rydberg state $|r\rangle$ is long: $\tau_r \gg 1/\Omega_0$.

Problem 3 Collective Rydberg blockade

Consider an ensemble of N atoms (ground states $|0\rangle_j$), each coupled to their Rydberg state $|r\rangle_j$ by a Rabi coupling Ω.

(3.a) Write down the Hamiltonian of this system.

(3.b) Assuming strong enough Rydberg interactions between all atoms, derive an effective two-level picture for the collective qubit states

$$|\tilde{0}\rangle = \prod_j |0\rangle_j, \qquad \text{and } |\tilde{1}\rangle = \frac{1}{\sqrt{N}} \sum_j |0...0r_j0...0\rangle. \tag{5}$$

Show that the collective Rabi frequency is

$$\Omega_N = \sqrt{N}\Omega. \tag{6}$$

Problem 4 Slowly-varying envelope approximation

The dynamics of the electric field operator $\hat{\boldsymbol{E}}(\boldsymbol{r}, t)$ is governed by the operator-valued Maxwell equation:

$$\left(\partial_t^2 - c^2 \boldsymbol{\nabla}^2\right) \hat{\boldsymbol{E}}(\boldsymbol{r}, t) = -\frac{1}{\epsilon_0} \partial_t^2 \hat{\boldsymbol{P}}(\boldsymbol{r}, t), \tag{7}$$

where $\hat{\boldsymbol{P}}(\boldsymbol{r}, t)$ is the polarization density operator.

(4.a) Show that the slowly varying fields $\hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{r}, t)$, defined by

$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) = \sqrt{\frac{\hbar\omega_0}{2\epsilon_0}} \hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{r}, t) e^{i(k_0 z - \omega_0 t)} \tag{8}$$

with $\omega_0 = ck_0$, and $\hat{\boldsymbol{\mathcal{P}}}(\boldsymbol{r}, t)$, defined by

$$\hat{\boldsymbol{P}}(\boldsymbol{r}, t) = \hat{\boldsymbol{\mathcal{P}}}(\boldsymbol{r}, t) e^{i(k_0 z - \omega_0 t)}, \tag{9}$$

satisfy the paraxial wave equation:

$$\left(\partial_t + c\partial_z - i\frac{c}{2k_0} \boldsymbol{\nabla}_\perp^2\right) \hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{r}, t) = \frac{i}{\hbar} \sqrt{\frac{\hbar\omega_0}{2\epsilon_0}} \hat{\boldsymbol{\mathcal{P}}}(\boldsymbol{r}, t) \tag{10}$$

Hint: keep only slowly varying terms and work in Fourier-space, both for time and space directions.

(4.b) Formulate the effective Hamiltonian $\hat{\mathcal{H}}_{\text{eff}}$ from which the paraxial wave equation (10) is obtained as follows:

$$i\hbar\partial_t \hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{r}, t) = [\hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{r}, t), \hat{\mathcal{H}}_{\text{eff}}]. \tag{11}$$

Note that $[\hat{\mathcal{E}}_\mu(\boldsymbol{r}), \hat{\mathcal{E}}_\nu^\dagger(\boldsymbol{r}')] = \delta_{\mu,\nu}\delta(\boldsymbol{r} - \boldsymbol{r}')$.