Problem Set 2:

Handout: Fri, May 08, 2020; Solutions: Fri, May 22, 2020

Problem 1 Regularized contact interactions

Solve the Lippmann-Schwinger equation to all orders for the contact interaction $V(\boldsymbol{r}) = g\delta^{(3)}(\boldsymbol{r})$.

(1.a) Show the following relation between the UV momentum cut-off Λ_0, the interaction strength g and the resulting scattering length a:

$$\frac{1}{g} = \frac{m_{\rm red}}{2\pi\hbar^2}\frac{1}{a} - \int^{\Lambda_0} d^3\boldsymbol{k}\ (2\pi)^{-3}\frac{m_{\rm red}}{\hbar^2 k^2}. \tag{1}$$

(1.b) Derive the expression for the effective range $r_{\rm eff}$, as a function of the UV momentum cut-off Λ_0:

$$r_{\rm eff} = \frac{\pi}{4}\Lambda_0^{-1}. \tag{2}$$

Hint: For $\epsilon \to 0^-$ it holds: $\int_0^1 dx\ [1-(k/x)^2 + i\epsilon]^{-1} = 1 + i\frac{\pi}{2}k - k^2 + \mathcal{O}(k^3)$.

(1.c) Now consider the 1D Lippmann-Schwinger equation, for 1D contact interactions $V(x) = g_{\rm 1D}\delta(x)$. Derive the relation between $g_{\rm 1D}$ and the scattering length a (defined in 1D by $f_k \to -1/[1+ika]$ for $k \to 0$; note that $f(k) = -T_E(k) i\pi 2m_{\rm red}/(|k|\hbar^2))$ – is UV regularization required in this case?

Problem 2 General solution of Feshbach resonances

Here we treat the intermediate steps in the lecture's derivation of Feshbach resonances.

(2.a) Show that the formal solution of the Lippmann-Schwinger eq. $\hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{T}$ is given by

$$\hat{T} = (1 - \hat{V}\hat{G}_0)^{-1}\hat{V}. \tag{3}$$

(2.b) Show that the solution (3) can also be written as:

$$\hat{T} = \hat{V}(1 - \hat{G}_0\hat{V})^{-1}. \tag{4}$$

(2.c) Consider the Feshbach T-matrix for the open channels $\hat{T}_0$ from p. II-41 of the lecture notes. Use the result from (2.a) to show that:

$$\hat{T}_0 = (E - \hat{\mathcal{H}}_0 + i\epsilon)(E + i\epsilon - \hat{\mathcal{H}}_0 - \hat{V})^{-1}\hat{V}. \tag{5}$$

(2.d) Use the identity $(\hat{A} - \hat{B})^{-1} = \hat{A}^{-1}[1 + \hat{B}(\hat{A} - \hat{B})^{-1}]$ to write (noting $\hat{V} = \hat{V}_0 + \hat{\mathcal{H}}'_{00}$):

$$(E + i\epsilon - \hat{\mathcal{H}}_0 - \hat{V})^{-1} = (E + i\epsilon - \hat{\mathcal{H}}_0 - \hat{V}_0)^{-1}\big[1 + \hat{\mathcal{H}}'_{00}(E + i\epsilon - \hat{\mathcal{H}}_0 - \hat{V})^{-1}\big]. \tag{6}$$

(2.e) Using (6), and (3) for the open-channel-only T-matrix $\hat{T}_0^{(0)}$, show that (5) can be written:

$$\hat{T}_0 = \hat{T}_0^{(0)} + (1 - \hat{V}_0\hat{G}_0)^{-1}\ \hat{\mathcal{H}}'_{00}\ (1 - \hat{G}_0\hat{V})^{-1}. \tag{7}$$

Problem 3 The $1/r^2$ potential

(3.a) Consider the following one-dimensional Schrödinger equation with a $1/r^2$ potential,

$$\left(-\partial_r^2 - \lambda r^{-2}\right)\psi_1(r) = \frac{2m}{\hbar^2} E_1 \ \psi_1(r). \tag{8}$$

From a given solution of this equation, $\psi_1(r)$ with eigenenergy E_1, construct re-scaled solutions $\psi_\alpha(r)$ with eigenenergies $E_\alpha \neq E_1$ for $\alpha \in \mathbb{R}_{>0}$ (– use scale-invariance!).

(3.b) Now consider the regularized $1/r^2$ potential, with $s_0 \in \mathbb{R}$ and $r_0 \in \mathbb{R}_{>0}$:

$$V(r) = -\frac{\hbar^2}{2m}\left(s_0^2 + \frac{1}{4}\right) \times \begin{cases} -\infty & r < r_0 \\ 1/r^2 & r \geq r_0 \end{cases}. \tag{9}$$

As discussed in the lecture, its bound states are given by $\psi_n(r) = \sqrt{\kappa_n r} K_{is_0}(\kappa_n r)$. Show in the scaling limit $r_0 \to 0$ that the solutions satisfy

$$\kappa_n r_0 = e^{-n\pi/s_0} 2e^{-\gamma}[1 + \mathcal{O}(s_0)], \qquad n \in \mathbb{Z}_{>0}. \tag{10}$$

Hint: The modified Bessel functions of second kind can be expanded as follows,

$$K_{is_0}(z) \approx -\sqrt{\frac{\pi}{s_0 \sinh(\pi s_0)}} \times \sin\left[s_0 \log(z/2) - \arg\Gamma(1 + is_0)\right], \qquad |z| \ll 1. \tag{11}$$

Problem 4 Bound states in separable potentials

Consider a 2-body problem (reduced mass m_r)

$$\left(-\hbar^2 \frac{\partial_r^2}{2m_r} + \hat{V}\right)|\psi\rangle = E|\psi\rangle \tag{12}$$

with an interaction which is *separable* in the relative coordinate r, i.e.:

$$\hat{V} = -\lambda |g\rangle\langle g|, \qquad \lambda \in \mathbb{R}_{>0}. \tag{13}$$

You may assume that $g(r) = \langle \boldsymbol{r}|g\rangle \in \mathbb{R}$ is real.

(4.a) Write Eq. (12) in momentum representation and show that the bound state solution with energy $E_0 = -\hbar^2\kappa_0^2/(2m_r)$ becomes

$$\psi(k) = C\frac{g(k)}{\kappa_0^2 + k^2}, \qquad g(k) = \langle \boldsymbol{k}|g\rangle. \tag{14}$$

Derive an expression for the constant C.

(4.b) Show that the binding energy E_0 is determined by the equation

$$\lambda \int d^3\boldsymbol{k} \ \frac{|g(k)|^2}{\kappa_0^2 + k^2} = \frac{\hbar^2}{2m_r} \tag{15}$$

(4.c) Consider the choice (Yamaguchi potential):

$$g(k) = (k^2 + \beta^2)^{-1}. \tag{16}$$

Calculate $g(r)$ and show that (up to normalization):

$$\psi(r) \simeq \left(\frac{e^{-\kappa_0 r}}{r} - \frac{e^{-\beta r}}{r} \right). \tag{17}$$

(4.d) Solve the equation from (4.b) to find the bound state energy E_0 for the Yamaguchi potential from (4.c).