Problem Set 1:

Handout: Thu, Apr. 23, 2020; Solutions: Fri, May 8, 2020

Problem 1 Quasiparticle residue: Bose polarons

(1.a) Consider the Bose polaron problem from the lecture: $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{int}}$:

$$\hat{\mathcal{H}}_0 = \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} + \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}; \tag{1}$$

Assume that the interaction Hamiltonian $\hat{\mathcal{H}}_{\text{int}}$ between the impurity ($\sum_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} = 1$) and the Bosons ($\hat{a}_{\boldsymbol{k}}$) is arbitrary but translationally invariant. Show for the interacting ground state $|\Psi^0_{\text{int}}\rangle \equiv |\psi(\boldsymbol{k}_{\text{min}})\rangle$, in which $\boldsymbol{k}_{\text{min}}$ is chosen to minimize $\langle \hat{\mathcal{H}} \rangle$, that

$$Z_0 \equiv Z_0(\boldsymbol{k}_{\text{min}}) = \sum_{\boldsymbol{j}} |\langle \Psi^0_{\text{int}} | \hat{c}^\dagger_{\boldsymbol{j}} | \psi_0 \rangle|^2. \tag{2}$$

– This expression is useful for calculating the quasiparticle weight Z numerically without full access to the total conserved momentum quantum number.

Problem 2 Quasiparticle residue: topological excitations

(2.a) Use the definition from the lecture of topological excitations (anyons) as *well-defined low-energy eigenstates of a (many-body) system, which cannot be created by any local operator.* Argue that anyons are no 'quasiparticles' in the strict sense (i.e. the quasiparticle residue $Z = 0$ vanishes) if we consider an infinite system.

Problem 3 Van-der-Waals interactions

(3.a) To describe van-der-Waals interactions between two Rydberg atoms in states $|njp\rangle$, consider a simplified model with dipole-dipole couplings only to the following two states $|n-1js\rangle$ and $|njs\rangle$ in each of the two atoms. Show that the interaction potential at large distances has the form

$$V_{\text{Ry}}(r) = +\frac{C_6}{r^6}, \tag{3}$$

with $C_6 > 0$. Derive an expression for C_6 and show that $C_6 \propto n^{11}$: you may use that the dipole-matrix elements scale as n^2 (as in a hydrogen atom) and that $E_n \propto -n^{-2}$.

Problem 4 Scattering theory

(4.a) Using Eq. (2) on p. II-8 in the script and the the expansion of the plane-wave e^{ikz} into Legendre polynomials, show that the scattering amplitude f can be expressed in terms of the phase shifts δ_ℓ as:

$$f(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1)(e^{i2\delta_\ell} - 1) P_\ell(\cos\theta). \tag{4}$$

(4.b) Use the result in (a) to show that $\delta_0 = -ka$, where a is the scattering length.