Avalanches

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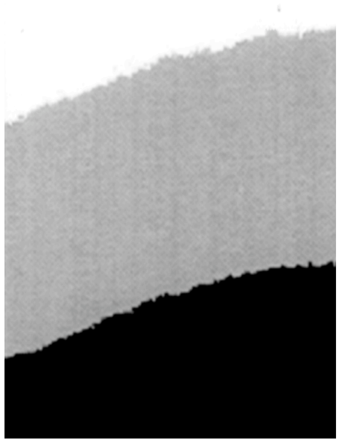
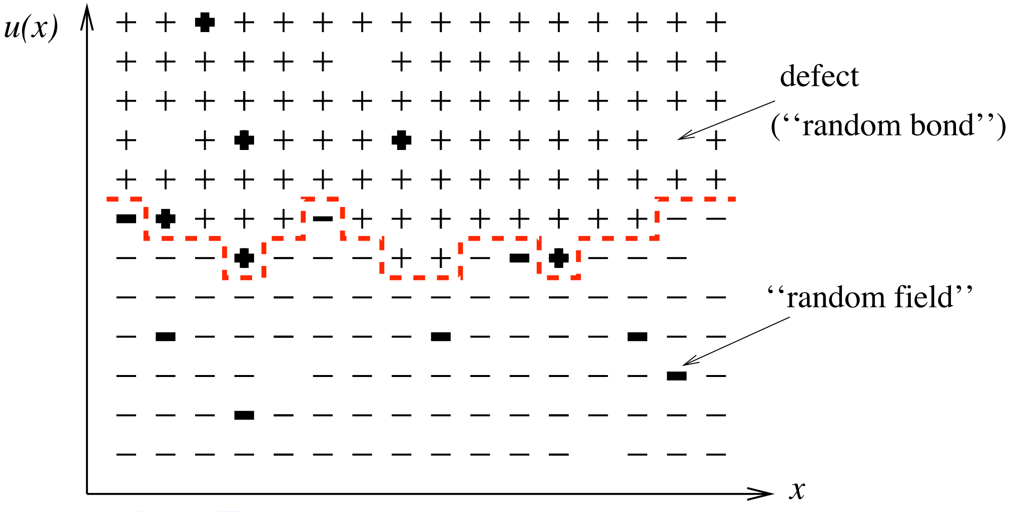
Muenchen, 2.5.2012

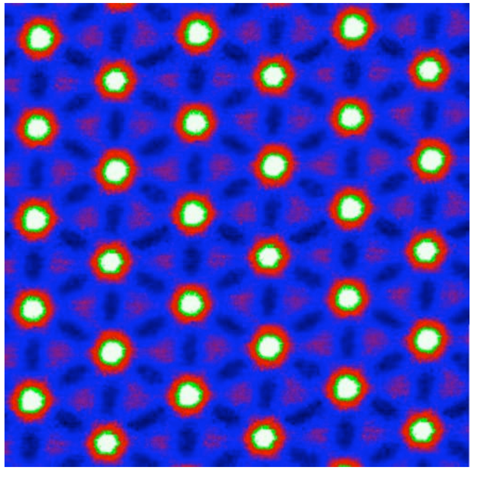
Physical Realizations

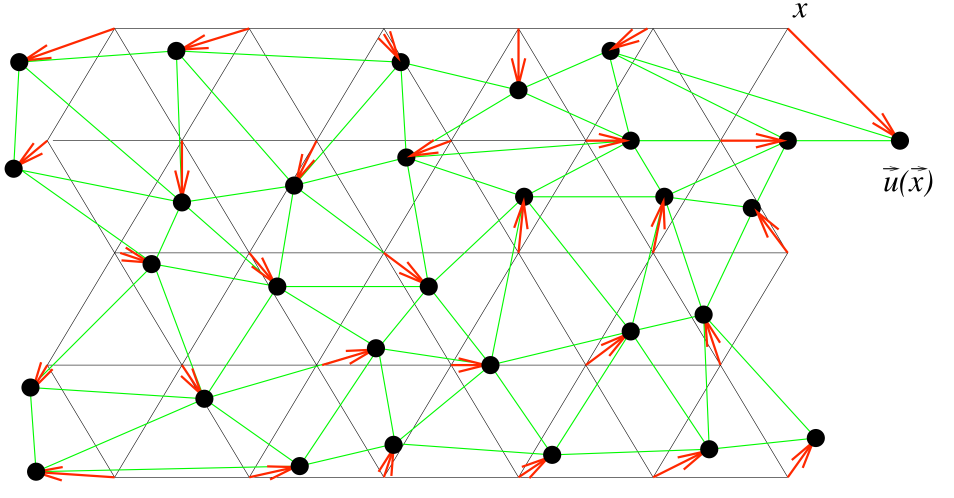
Domain-walls in magnets, temperature $T \to 0$ (Barkhausen noise)





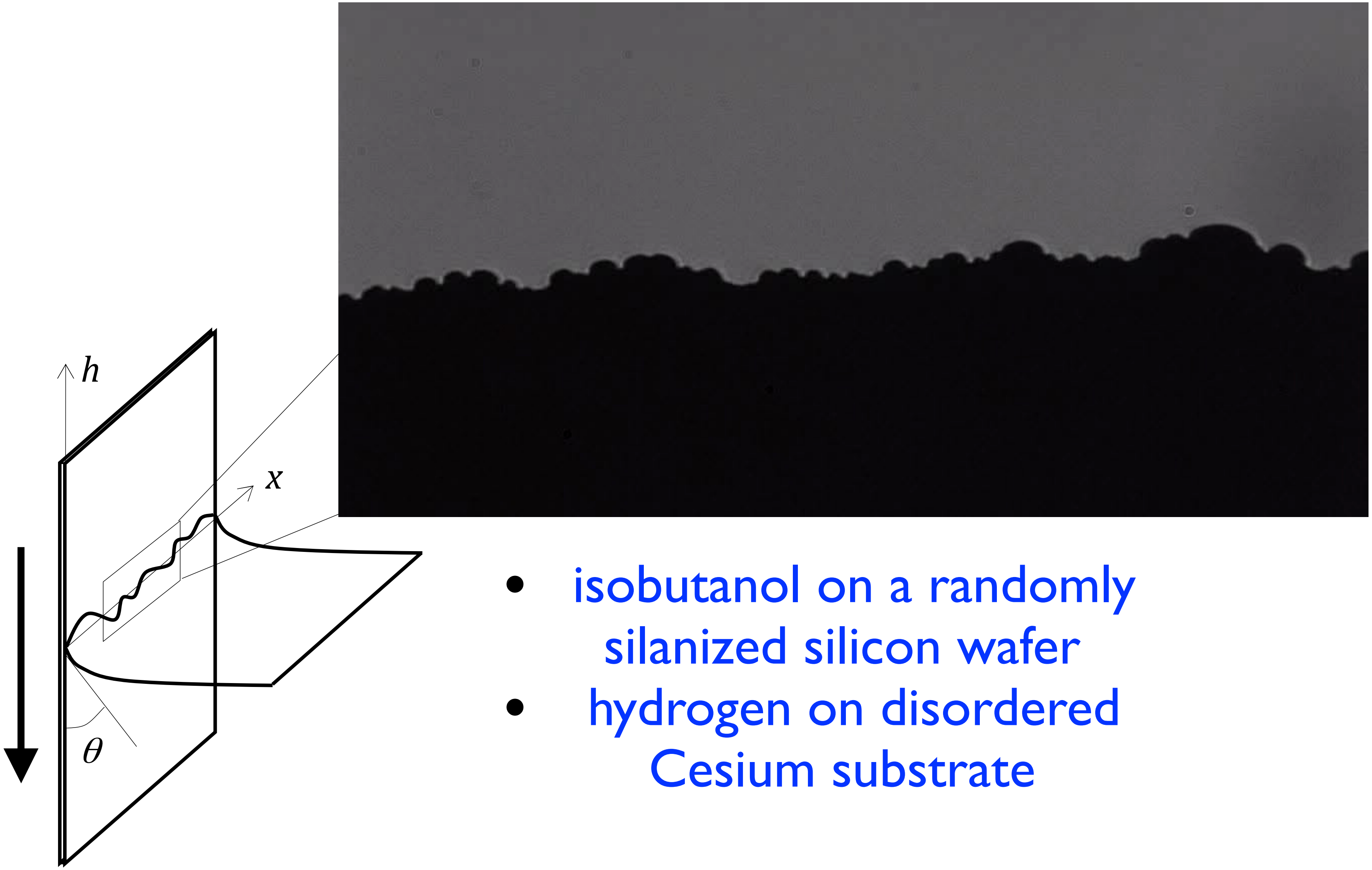
Vortex-lattice/Bragg glass





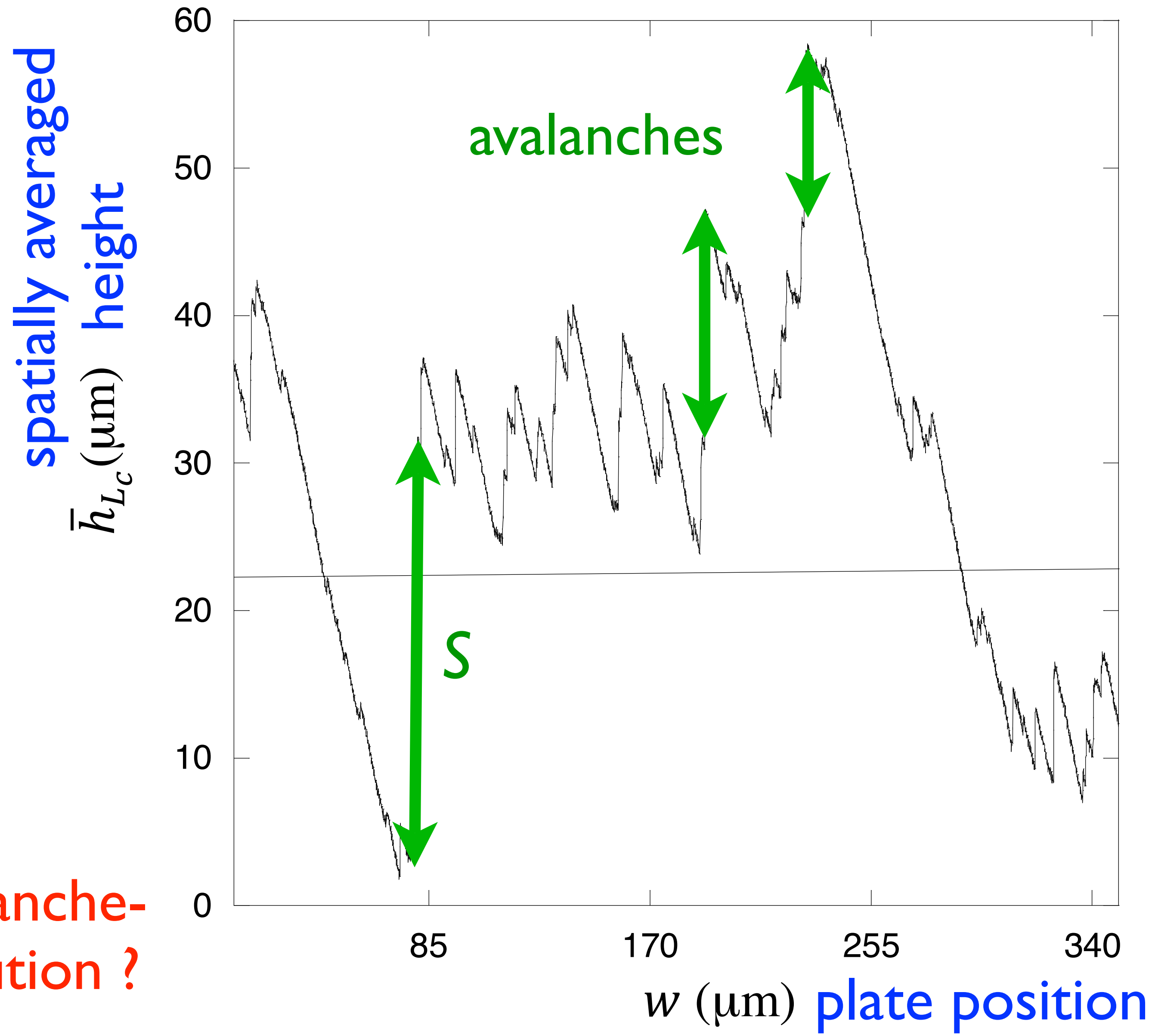
cracks - earthquakes - fracture - contact-line wetting

Contact line wetting



- isobutanol on a randomly silanized silicon wafer
- hydrogen on disordered Cesium substrate

height jumps = avalanches



what is avalanche-size distribution ?

The model

Displacement field $\qquad x \in \mathbb{R} \longrightarrow u(x) \in \mathbb{R}$

Elastic energy:
$$\mathscr{H}_{\text{el}} = \frac{1}{2} \int \frac{\mathrm{d}^d k}{2\pi} \, |\tilde{u}_k|^2 \, \varepsilon_k + \int_x \frac{m^2}{2} [u(x) - w]^2$$

for contact angle $\theta = 90°$: $\qquad \varepsilon_k \approx \sqrt{k^2 + \kappa^2} - \kappa$

$\kappa^{-1} = m^{-2}$ kapillary length $\qquad$ (instead of $\varepsilon_k = k^2$)

Disorder energy
$$\mathscr{H}_{\text{DO}} = \int \mathrm{d}^d x V(x, u(x))$$

with correlations
$$\overline{V(x,u)V(x',u')} = \delta^d(x - x')R(u - u')$$

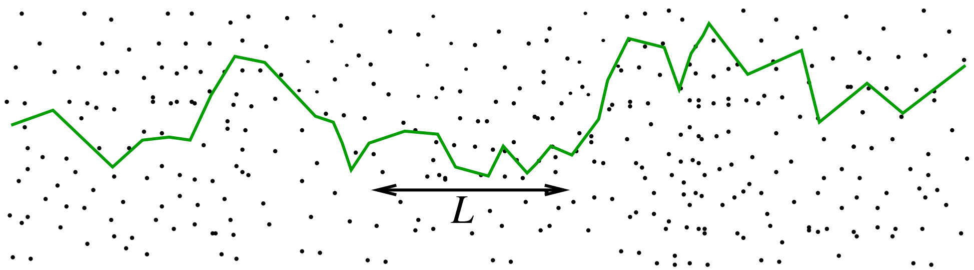
Simple theory for zero temperature $T = 0$

Suppose $R(u)$ is analytic. Then to all orders in perturbation theory:

$$\left\langle [u(x) - u(0)]^2 \right\rangle \sim -R''(0) x^{4-d} + O(T)$$

shift in dimension by two from thermal 2-point function
$\langle [u(x) - u(0)]^2 \rangle = T x^{2-d}$: dimensional reduction.

Experimentally wrong beyond Larkin length:



elastic energy $\mathcal{E}_{\text{el}} = c L^{d-2}$

disorder $\mathcal{E}_{\text{DO}} = \bar{f} \left(\frac{L}{r}\right)^{d/2}$

$$\mathcal{E}_{\text{el}} = \mathcal{E}_{\text{DO}} \quad \Rightarrow \quad L_c = \left(\frac{c^2}{\bar{f}^2} r^d\right)^{\frac{1}{4-d}}$$

critical dimension is $d_c = 4$

u dimensionless in $d_c = 4$ $\Rightarrow$ all powers of u relevant!

Need functional RG!

Old idea: Wegner, Houghton (1973)

for disordered systems: D.S. Fisher (1985)

Functional renormalization group (FRG)

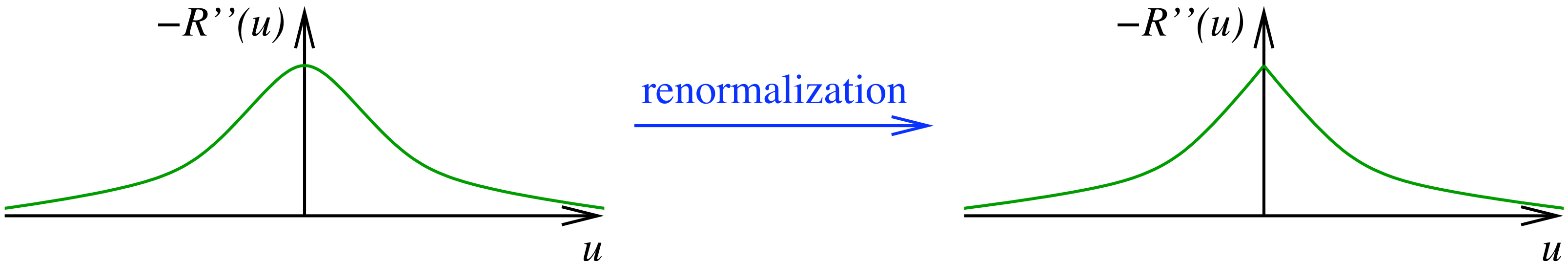
(D. Fisher 1986)

$$\frac{\mathscr{H}[u]}{T} = \frac{1}{2T}\sum_{\alpha=1}^{n}\left[\int_k \varepsilon_k |\tilde{u}_k^{\alpha}|^2 + \int_x m^2(u^{\alpha}(x) - w)^2\right]$$
$$-\frac{1}{2T^2}\int_x \sum_{\alpha,\beta=1}^{n} R\big(u^{\alpha}(x) - u^{\beta}(x)\big)$$

Functional renormalization group equation (FRG) for the disorder correlator $R(u)$ at 1-loop order:

$$-\frac{m\mathrm{d}}{\mathrm{d}m}R(u) = (\varepsilon - 4\zeta)R(u) + \zeta u R'(u) + \frac{1}{2}R''(u)^2 - R''(u)R''(0)$$

Solution for force-force correlator $-R''(u)$:



Cusp: $R''''(0) = \infty$ appears after finite RG-time (at Larkin-length)

FRG at 2-loop order

$$\partial_\ell R(u) = (\varepsilon - 4\zeta)R(u) + \zeta u R'(u) + \frac{1}{2}R''(u)^2 - R''(u)R''(0)$$
$$+\frac{1}{2}\left[R''(u) - R''(0)\right]R'''(u)^2 + \lambda\frac{1}{2}R'''(0^+)^2 R''(u)$$

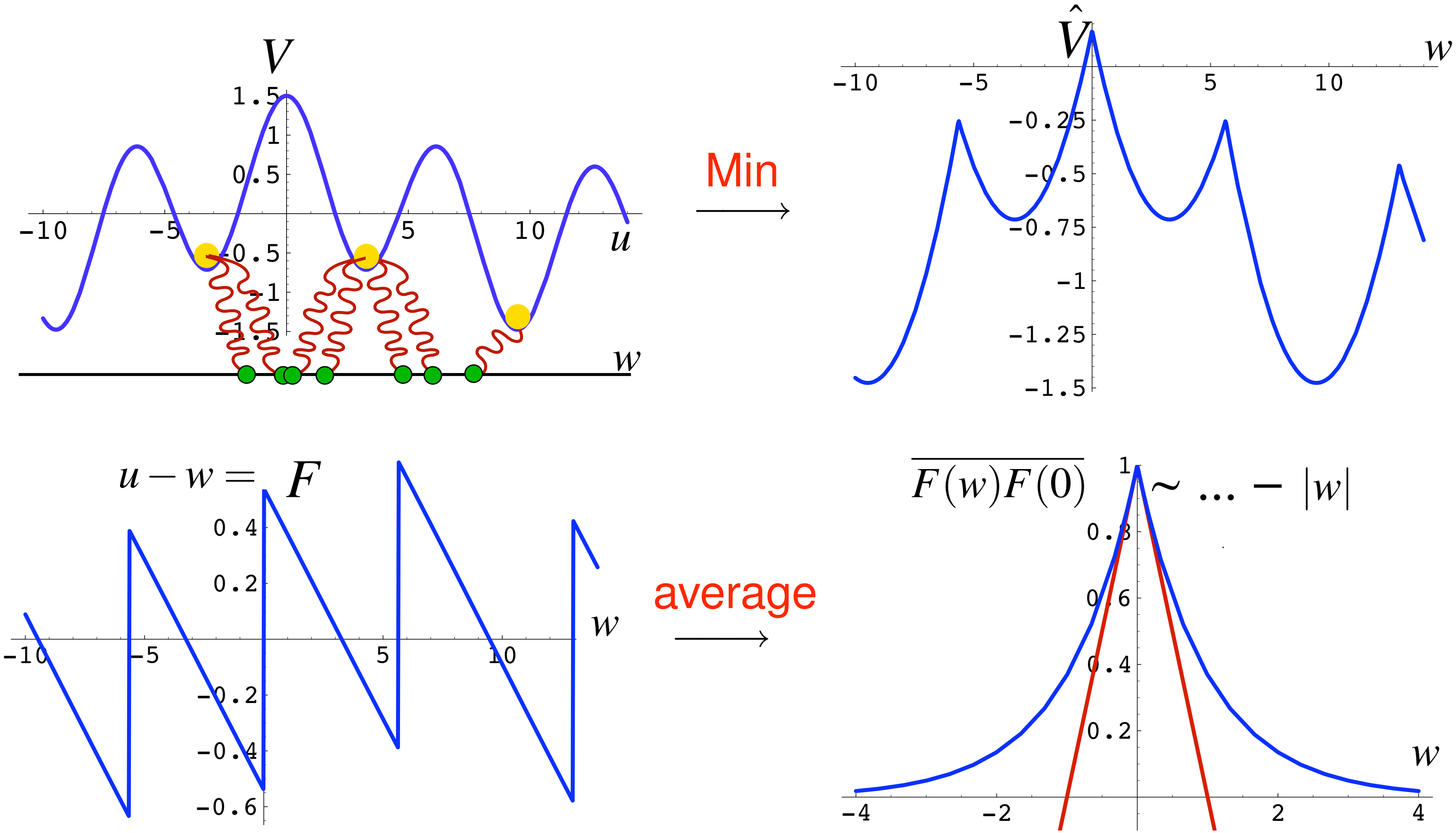
$\lambda = -1$ statics, $\lambda = 1$ (depinning)

Universality classes

- periodic disorder
- random field disorder: $\Delta(u) = -R''(u)$ short-ranged
  statics: $\zeta = \frac{\varepsilon}{3}$ (exact), depinning $\zeta = \frac{\varepsilon}{3}(1 + 0.14331\varepsilon + \ldots)$
- random bond: $R(u)$ short-ranged
  statics: $\zeta = 0.20829804\varepsilon + 0.006858\varepsilon^2$, dynamics $\rightarrow$ RF

Why is a cusp necessary?

...calculate effective action for single degree of freedom...



Renormalized Disorder Correlator in FRG

$$\mathcal{H}^w[u] = \int \frac{1}{2}[\nabla u(x)]^2 + V\big(x, u(x)\big) + \frac{m^2}{2}[u(x) - w]^2 \, \mathrm{d}^d x$$

Local minimum $u_w(x)$ satisfies:

$$0 = \frac{\delta \mathcal{H}^w[u]}{\delta u_w(x)} = -\nabla^2 u_w(x) - F\big(x, u_w(x)\big) + m^2\,[u_w(x) - w]$$

Center-of-mass u_w fluctuates around w

$$u_w - w := \frac{1}{L^d}\int [u_w(x) - w]\,\mathrm{d}^d x = \frac{1}{L^d m^2}\int F\big(x, u_w(x)\big)\,\mathrm{d}^d x$$

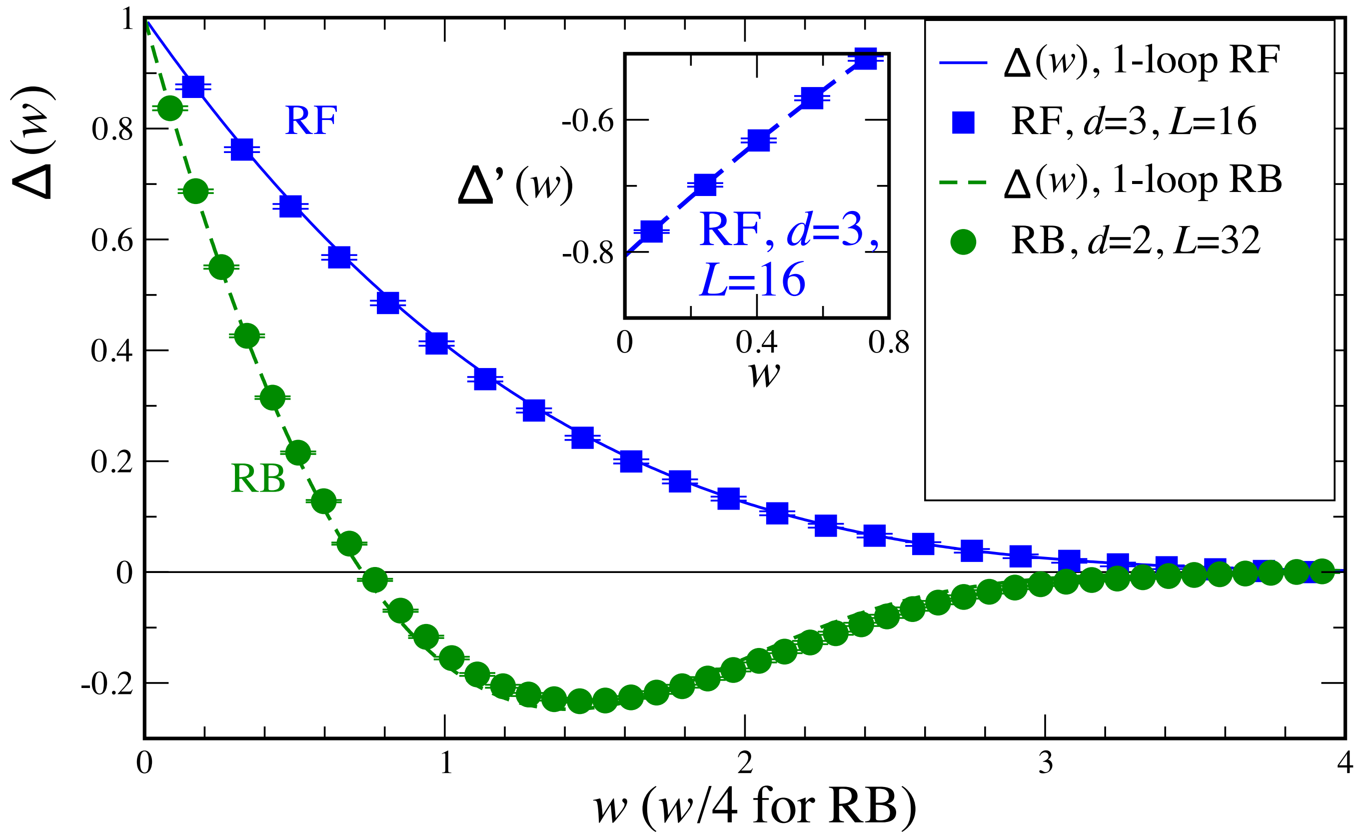
Thus naively

$$\overline{h_w h_{w'}} = \overline{[u_w - w]\,[u_{w'} - w']} = \frac{\Delta(w - w')}{L^d m^4}$$

FRG - Legendre-transform ... confirm this picture !

Measuring the cusp = effective action

PLD+KW+A. Middleton

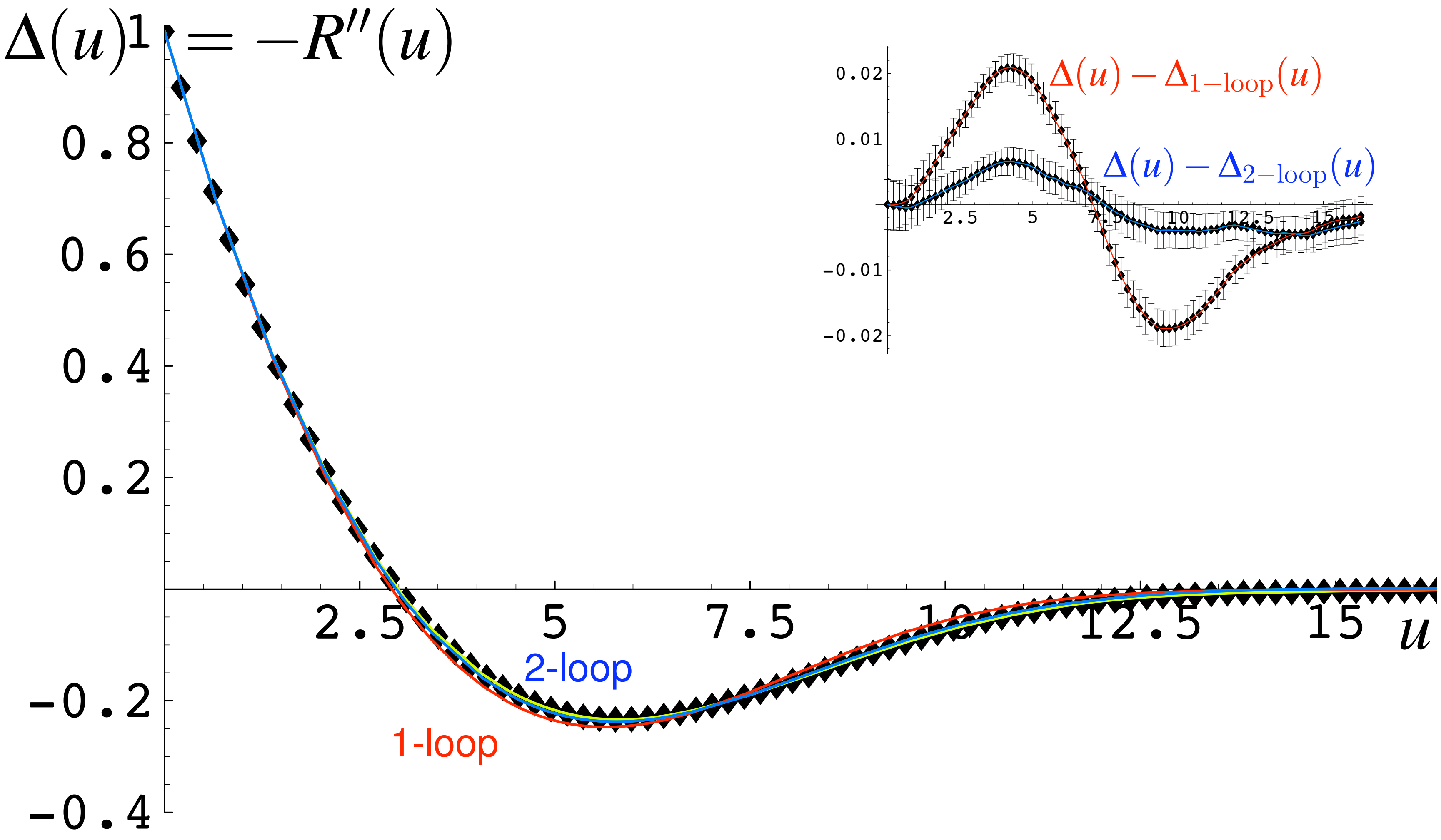


$$\Delta(w - w') = m^4 L^d \overline{[u_w - w]\,[u_{w'} - w']}$$

Δ = renormalized disorder correlator

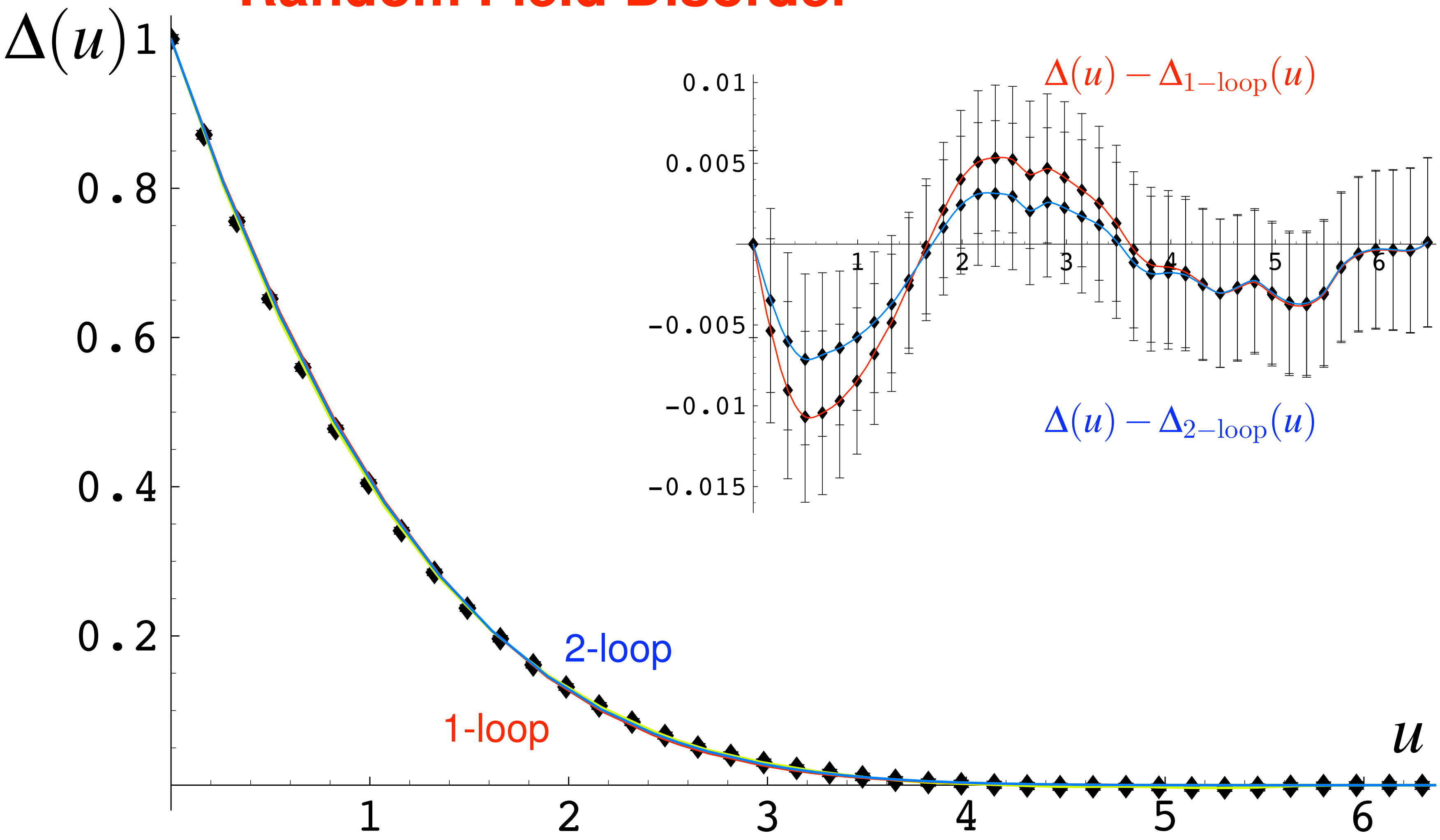
Random Bond (short-range correlated potential), $d = 1$

$\zeta = 0.208298\varepsilon + 0.006858\varepsilon^2$: 0.625 (1 loop), 0.687 (2 loop), $2/3$ (exact).



A. Middleton, P. Le Doussal, KW, PRL 98 (2007) 155701

Random Field Disorder

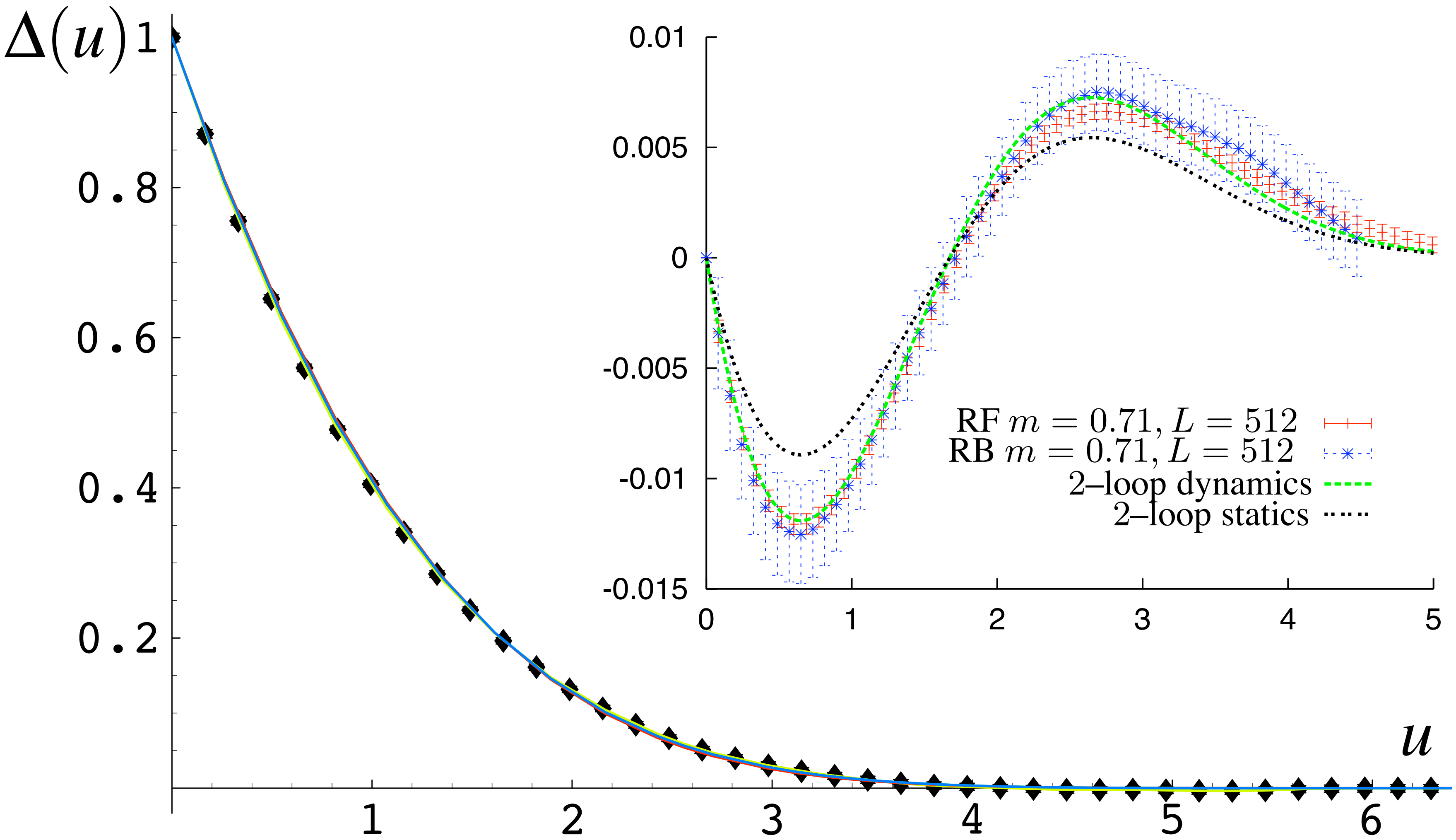


Interface in RF Ising model in $2+1$ dimensions ($d = 2$, $\varepsilon = 2$, $N = 1$)

A. Middleton, P. Le Doussal, KW, PRL 98 (2007) 155701

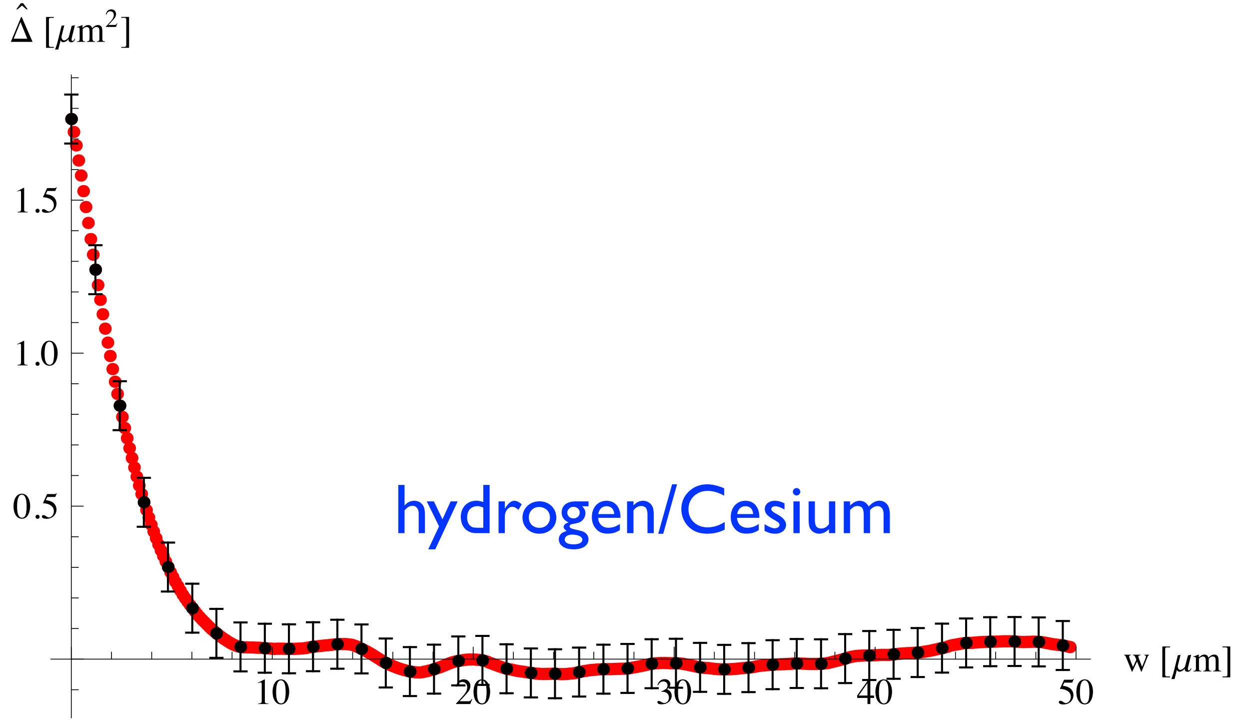
Depinning in 1+1 dimensions

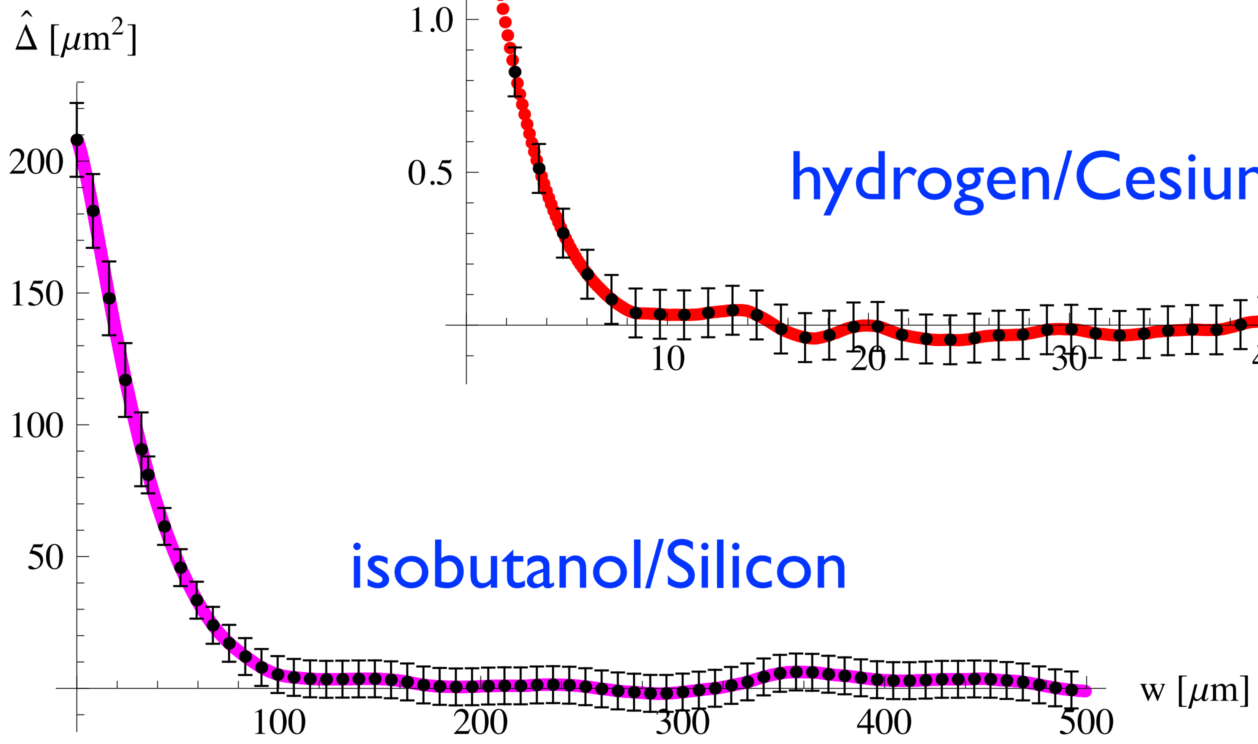
$\zeta = \frac{\varepsilon}{3} + 0.04777\varepsilon^2$: 1.0 (1 loop), 1.2 ± 0.2 (2 loop), 1.25 (numerics).



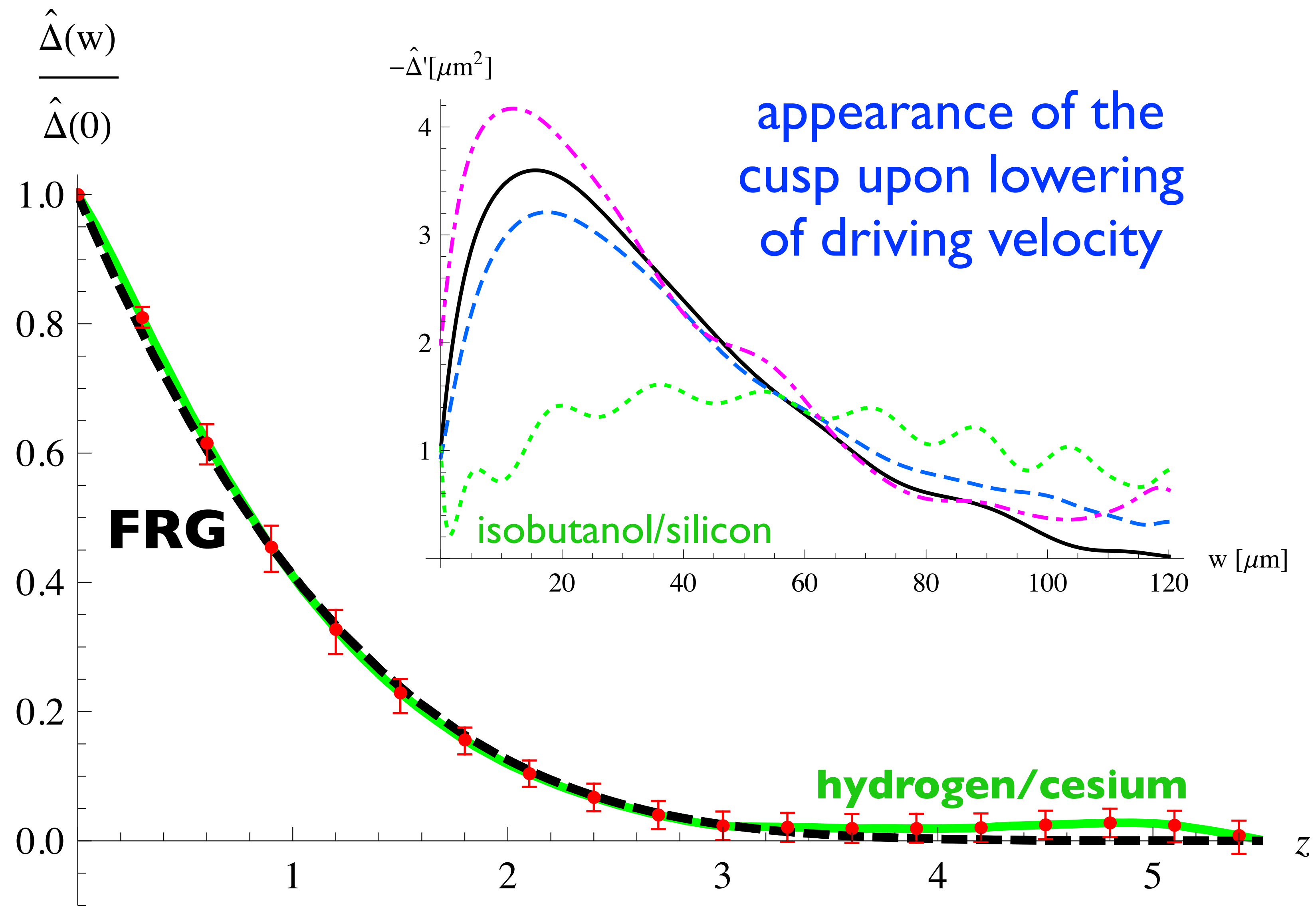
A. Rosso, P. Le Doussal, KW, PRB 75 (2007) 220201

Experiments on contact line





The renormalized force-force correlator

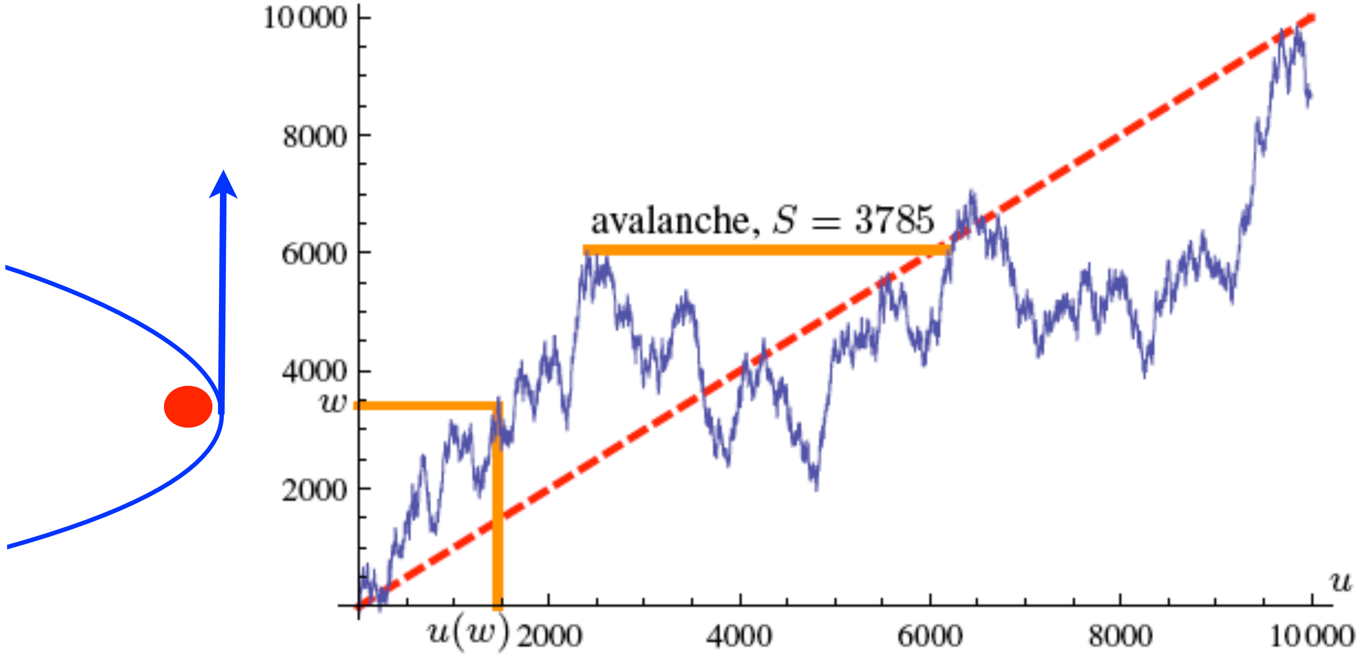


Avalanches

- avalanches appear in many systems: contact-lines, vortex lattices, domain walls, earthquakes, etc.
- Self-Organized Criticality (SOC)
- Abelian Sandpile Model (ASM) is best-known example
- ASM is equivalent to:
  - uniform spanning trees (UST)
  - loop-erased random walks (LERW)
- Mean-Field (MF) treatment available (Galton process)
- conjecture by Middleton-Narayan that Charge-Density Waves (CDW) are equivalent to ASM
  - leads to field-theory conjecture (Fedorenko, Le Doussal Wiese), with predictions for sub-leading logs in d=4.
  - recently checked in numerical simulations by Grassberger
- Decaying Burgers turbulence

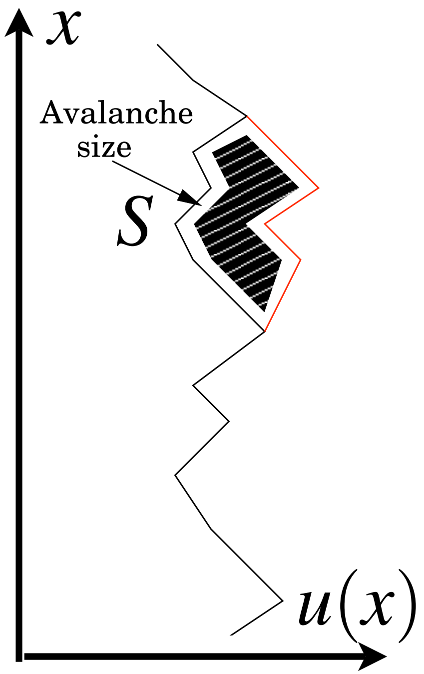
The Galton process

- old quesstion: survival probability of male line (Galton, Watson1873)
- equivalent: driven particle in random force landscape which itself is a Brownian = records with drift



$$P(S) \sim S^{-3/2} \mathrm{e}^{-S/S_m}$$

Slope at the cusp and avalanche size moments



$$\rho \langle S \rangle |w - w'| = L^d \overline{|u_w - u_{w'}|} = L^d |w - w'|$$

#avalanches/unit length

$$\rho \langle S^2 \rangle |w - w'| \approx L^{2d} \overline{|u_w - u_{w'}|^2}$$
$$\approx 2L^d \frac{|\Delta'(0^+)|}{m^4} |w - w'|$$

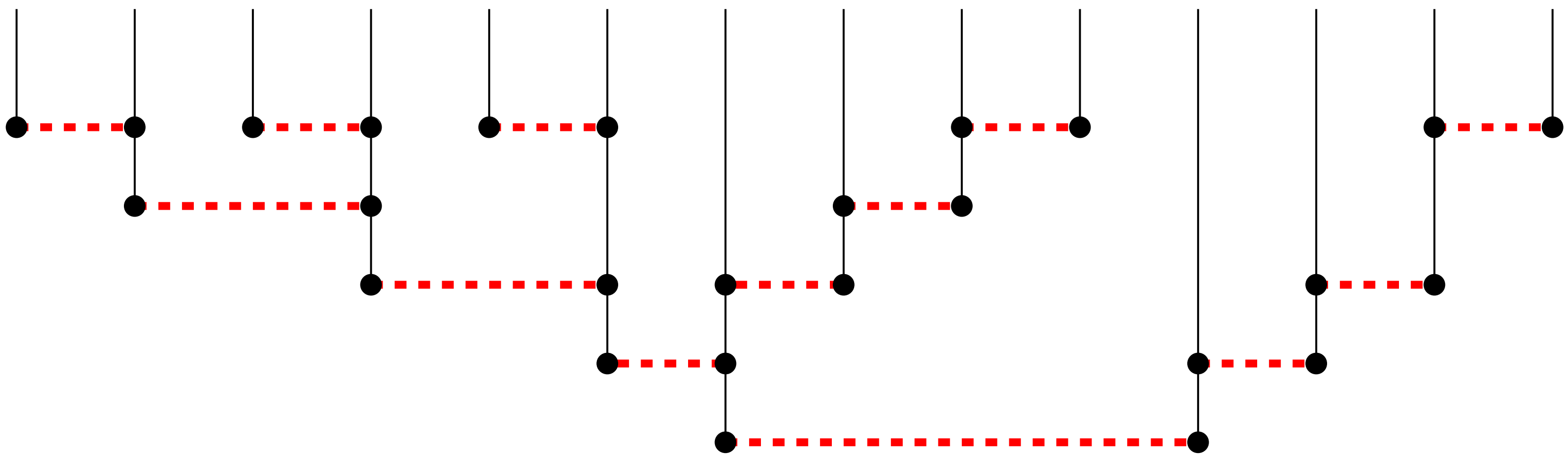
together:
(exact)

$$S_m := \frac{\langle S^2 \rangle}{2 \langle S \rangle} = \frac{|\Delta'(0^+)|}{m^4}$$

Generating function for avalanche size moments

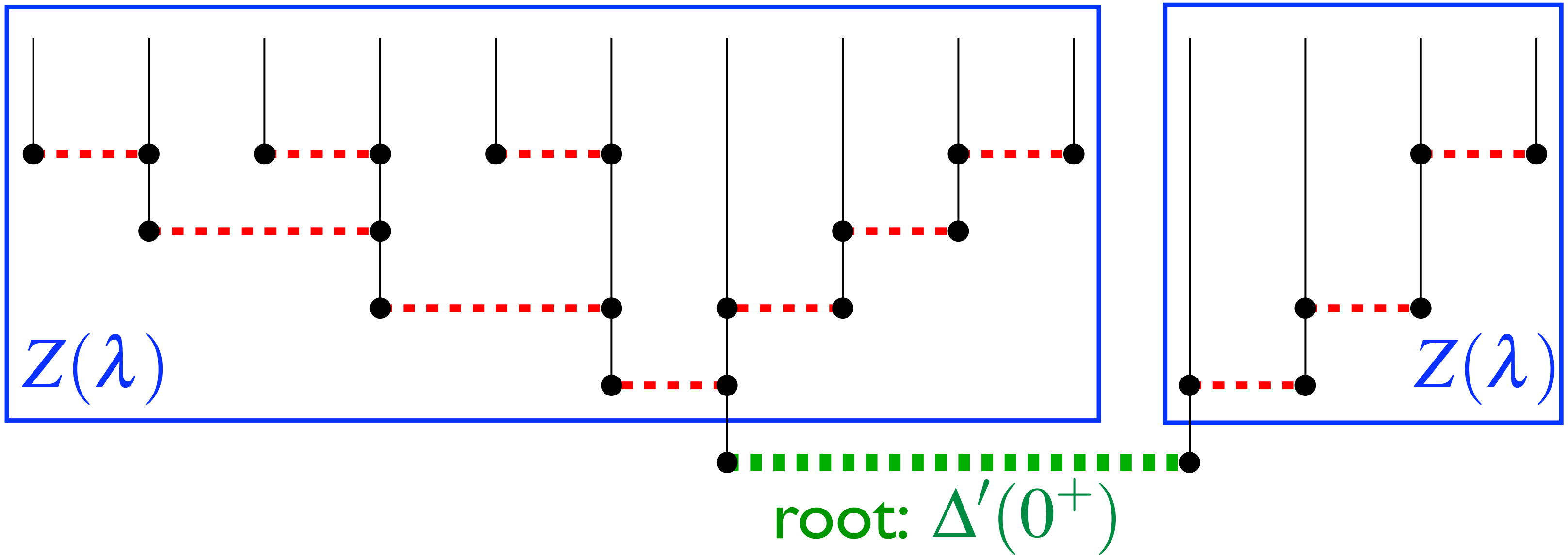
$$Z(\lambda) = \frac{1}{\langle S \rangle} \left(\langle e^{\lambda S} \rangle - 1 - \lambda \langle S \rangle \right)$$

$$\overline{e^{\lambda [u(w) - w - u(0)]}} - 1 = Z(\lambda) w + O(w^2) \quad \text{for} \quad w > 0$$



Tree resummation (I)

Rooted trees:



Resummation: $$Z(\lambda) = \lambda - \Delta'(0^+) Z(\lambda)^2$$

sufficient for N=1 avalanche-size distribution

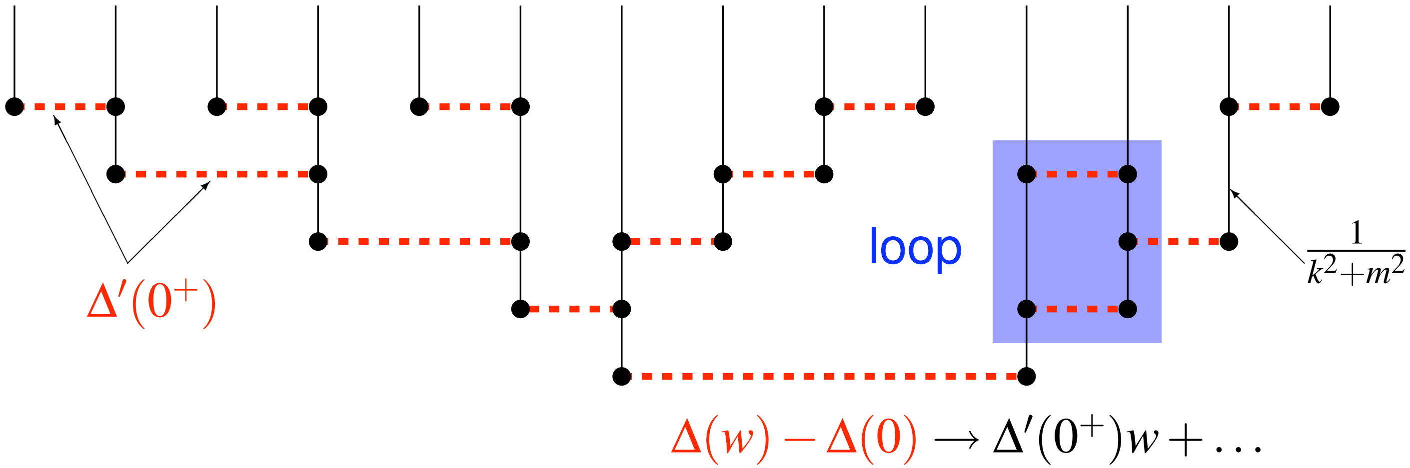
FRG-calculation

calculate the generating function $Z(\lambda)$ of avalanche-sizes S:

$$Z(\lambda) = \frac{1}{\langle S \rangle}\left(\langle e^{\lambda S}\rangle - 1 - \lambda \langle S \rangle\right)$$

$$\overline{e^{\lambda[u(w)-w-u(0)]}} - 1 = Z(\lambda) w + O(w^2) \quad \text{for} \quad w > 0\,.$$

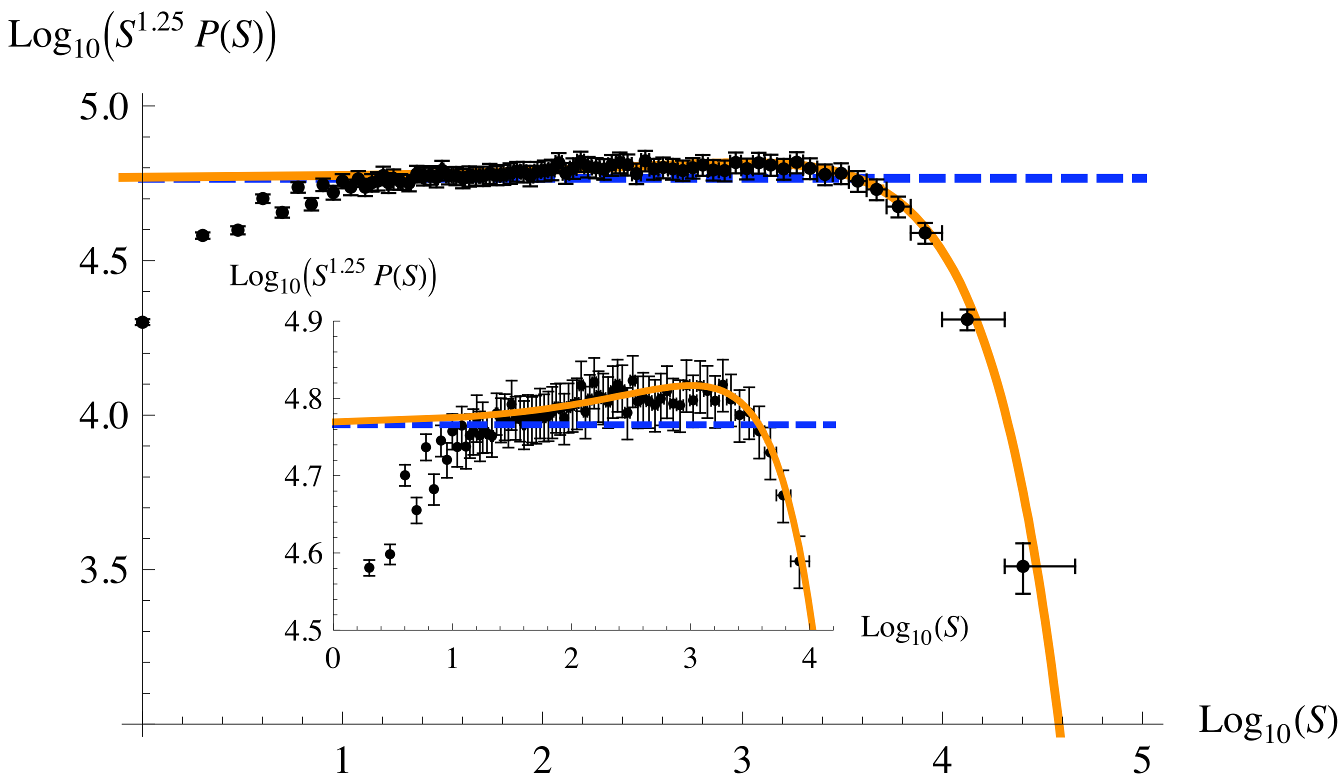
$$Z(\lambda) = \sum$$



Recursion Relation:

$$Z(\lambda) = \lambda - \underbrace{\Delta'(0^+) Z(\lambda)^2}_{\text{trees}} + \frac{\Delta''(0)}{\Delta'(0^+)} \sum_{n\geq 3} (n+1) 2^{n-2} \int_k \underbrace{\frac{[-\Delta'(0^+) Z(\lambda)]^n}{(k^2+1)^n}}_{\text{loops with } n \text{ outgoing legs}}\,,$$

Avalanche distribution



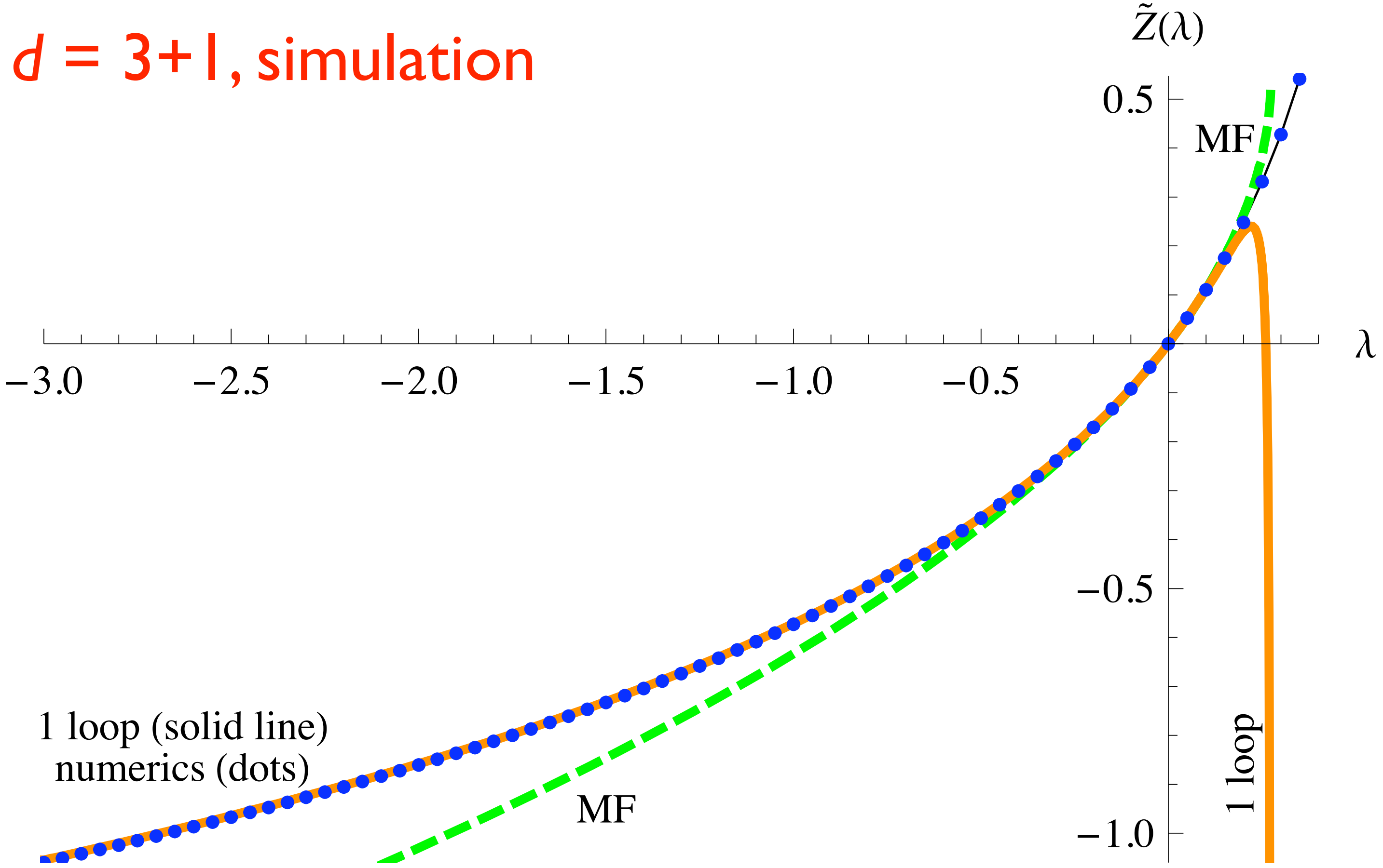
$$P(S) = \frac{\langle S \rangle}{2\sqrt{\pi}} S_m^{\tau-2} A S^{-\tau} \exp\left(C\sqrt{\frac{S}{S_m}} - \frac{B}{4}\left[\frac{S}{S_m}\right]^{\delta} \right)$$

$$\tau = \frac{3}{2} - \frac{1}{8}(\varepsilon - \zeta) + \dots$$

$$\delta = 1 + \frac{1}{4}(\varepsilon - \zeta) + \dots$$

Numerical results for RF interface in $d = 3+1$
P. Le Doussal, A. Middleton, KW
arXiv:0803.1142

RF, d = 3+1, simulation

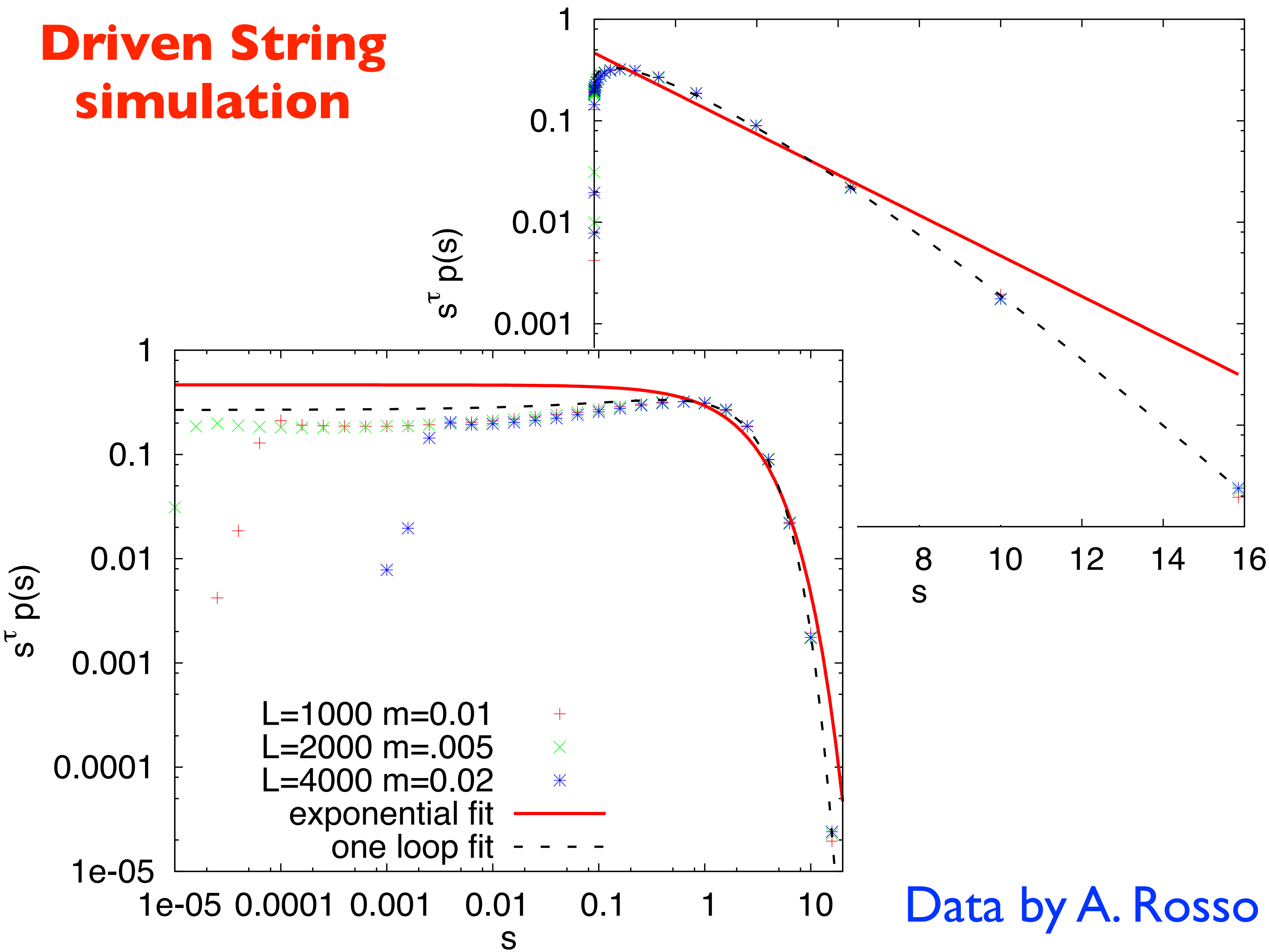


MF = trees

$$Z(\lambda) = \frac{1}{2}\left[1 - \sqrt{1-4\lambda}\right]$$
$$- \frac{\Delta''(0)}{4\sqrt{1-4\lambda}}\left[\log(1-4\lambda)(3\lambda + \sqrt{1-4\lambda} - 1) - 2(2\lambda + \sqrt{1-4\lambda} - 1)\right] + \dots$$

1 loop

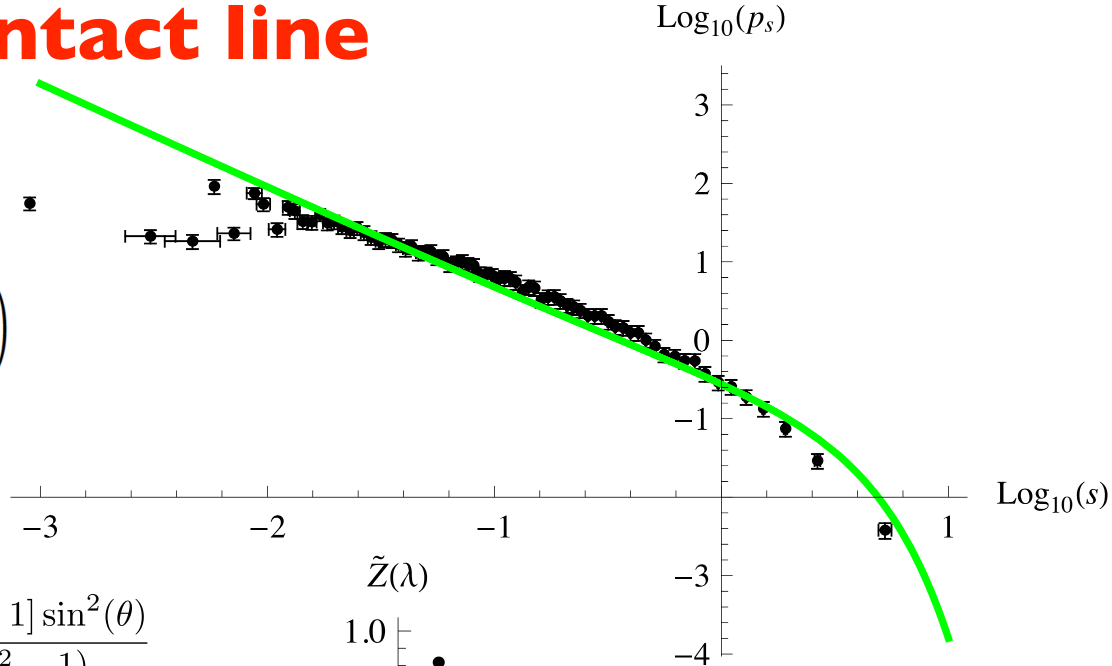
Driven String simulation



Data by A. Rosso

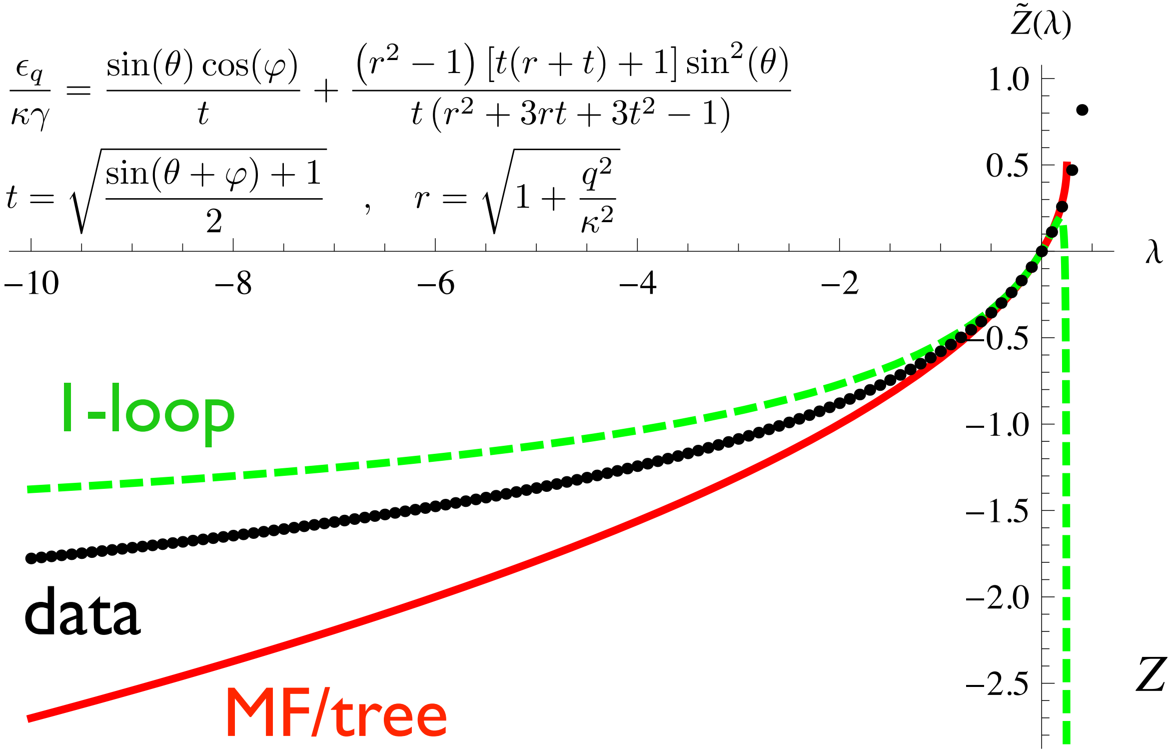
Experiments contact line

$$P(S) = A' \frac{\langle S \rangle}{2\sqrt{\pi}} S_m^{-2} \left[\left(\frac{S}{S_m} \right)^{-\tau} + D' \right] \times \exp\left(C' \sqrt{\frac{S}{S_m}} - \frac{B'}{4} \left[\frac{S}{S_m} \right]^{\delta'} \right)$$



$$\frac{\epsilon_q}{\kappa\gamma} = \frac{\sin(\theta)\cos(\varphi)}{t} + \frac{(r^2-1)\left[t(r+t)+1\right]\sin^2(\theta)}{t\,(r^2+3rt+3t^2-1)}$$

$$t = \sqrt{\frac{\sin(\theta+\varphi)+1}{2}} \quad , \quad r = \sqrt{1+\frac{q^2}{\kappa^2}}$$



$$Z(\lambda) := \int_0^\infty \mathrm{d}S\, p(S) \left[\mathrm{e}^{\lambda S} - 1 \right]$$

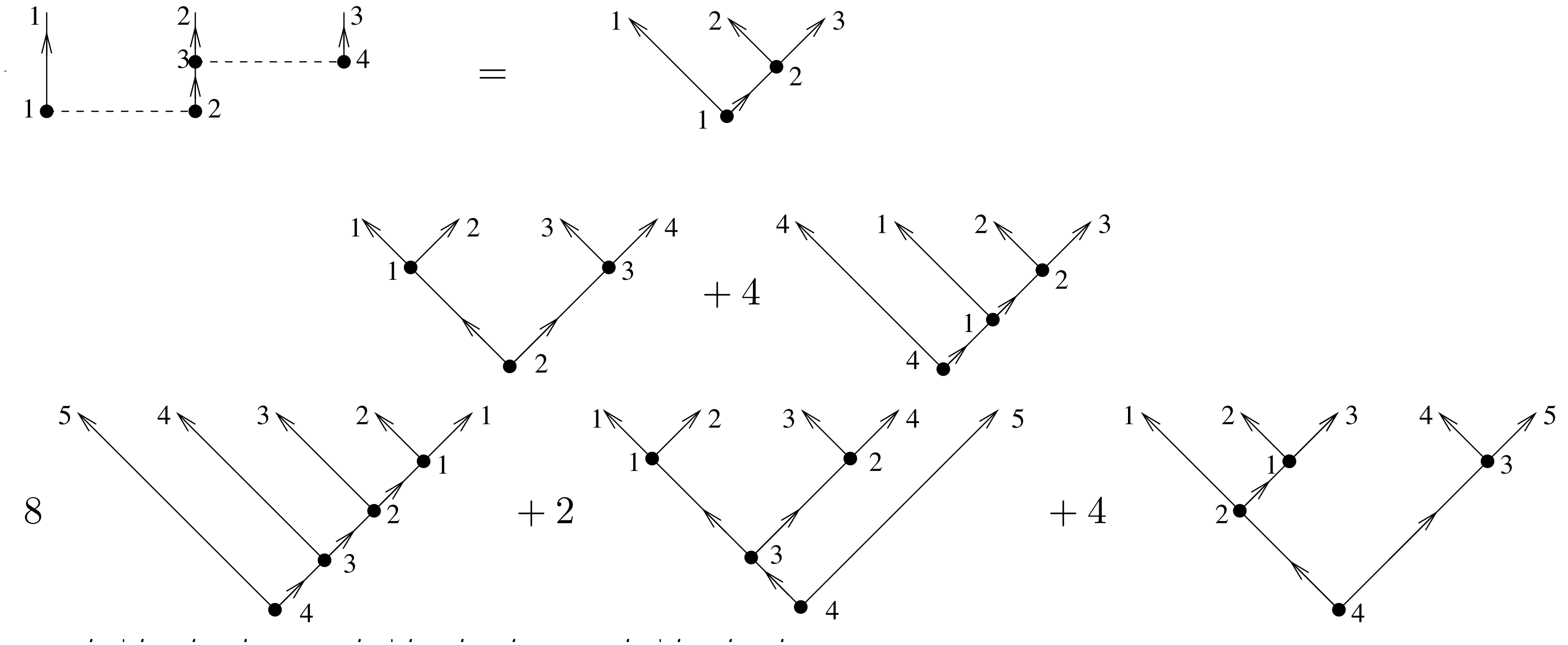
Velocity distribution in an avalanche

classical Langevin equation

$$\eta \partial_t u(x,t) = -\frac{\delta \mathscr{H}[u(t)]}{\delta u(t)}$$

$$= \nabla^2 u(x,t) + m^2 [w - u(x,t)] - \partial_u V(x, u(x,t))$$

moments are again trees...



A little field theory

$$S_0 = \int_{xt} \tilde{u}_{xt}(\eta\partial_t - \nabla_x^2 + m^2)\dot{u}_{xt}$$

$$S_{\mathrm{dis}} = -\frac{1}{2}\int_{xtt'} \tilde{u}_{xt}\tilde{u}_{xt'}\partial_t\partial_{t'}\Delta(v(t-t') + u_{xt} - u_{xt'})$$

Disorder Vertex

$$\begin{aligned} &\partial_t\partial_{t'}\Delta(v(t-t') + u_{xt} - u_{xt'}) \\ &= (v + \dot{u}_{xt})\partial_{t'}\Delta'(v(t-t') + u_{xt} - u_{xt'}) \\ &= (v + \dot{u}_{xt})\Delta'(0^+)\partial_{t'}\mathrm{sgn}(t-t') + \dots \end{aligned}$$

simplifies to

$$S_{\mathrm{dis}}^{\mathrm{tree}} = \Delta'(0^+)\int_{xt} \tilde{u}_{xt}\tilde{u}_{xt}(v + \dot{u}_{xt})$$

!!! simple cubic theory !!!

Avalanche Instanton

If $\lambda(x,t) = \lambda\,\delta(t)$ then the instanton equation is

$$(\partial_t - 1)\tilde{u}_t + \tilde{u}_t^2 = -\lambda\delta(t)$$

Solution

$$\tilde{u}_t = \frac{\lambda}{\lambda + (1-\lambda)e^{-t}}\theta(-t)$$

$$Z_{\text{tree}}(\lambda) = \left\langle \mathrm{e}^{\lambda \dot{u}(t)} - 1 \right\rangle\Big|_{t=0} = \int_{t<0} \tilde{u}_t = -\ln(1-\lambda)$$

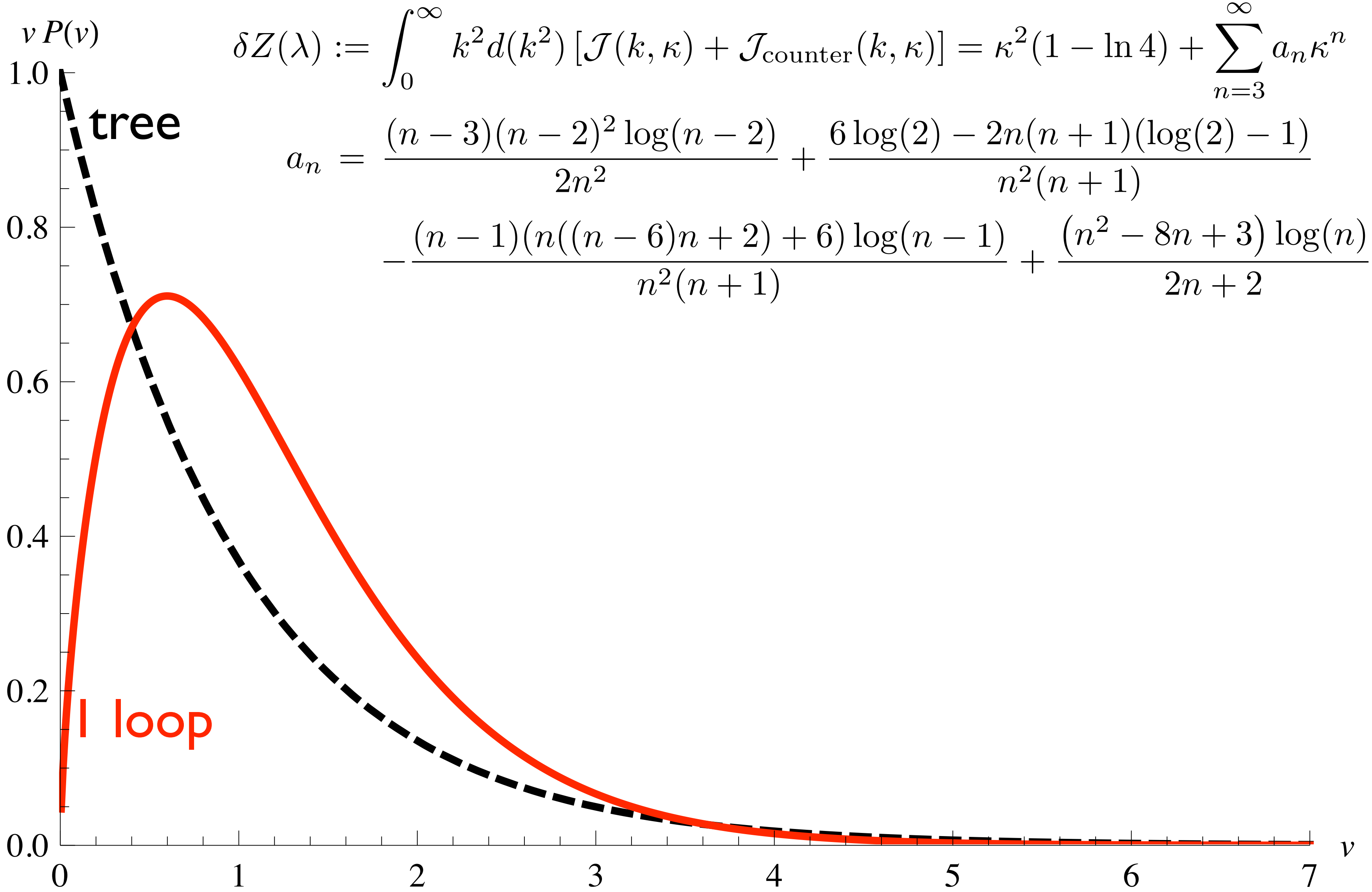
$$\mathscr{P}_{\text{tree}}(\dot{u}) = \frac{\mathrm{e}^{-\dot{u}}}{\dot{u}}$$

higher-point functions also possible.

Velocity distribution in avalanche: tree + loops

$$\delta Z(\lambda) := \int_0^\infty k^2 d(k^2)\, [\mathcal{J}(k,\kappa) + \mathcal{J}_{\text{counter}}(k,\kappa)] = \kappa^2(1-\ln 4) + \sum_{n=3}^{\infty} a_n \kappa^n$$

$$a_n = \frac{(n-3)(n-2)^2\log(n-2)}{2n^2} + \frac{6\log(2) - 2n(n+1)(\log(2)-1)}{n^2(n+1)} - \frac{(n-1)(n((n-6)n+2)+6)\log(n-1)}{n^2(n+1)} + \frac{\left(n^2-8n+3\right)\log(n)}{2n+2}$$



Decaying Burgers

$$\partial_t \vec{v}(\vec{r},t) + [\vec{v}(\vec{r},t)\cdot\nabla]\, \vec{v}(\vec{r},t) = \nu \nabla^2 \vec{v}(\vec{r},t)$$

$\uparrow$

inviscid limit $\nu \to 0$

curl-free velocity field: $\vec{v}(\vec{r},t) = \nabla V(\vec{r},t)$

in practice: specify $V(\vec{r},t=0)$

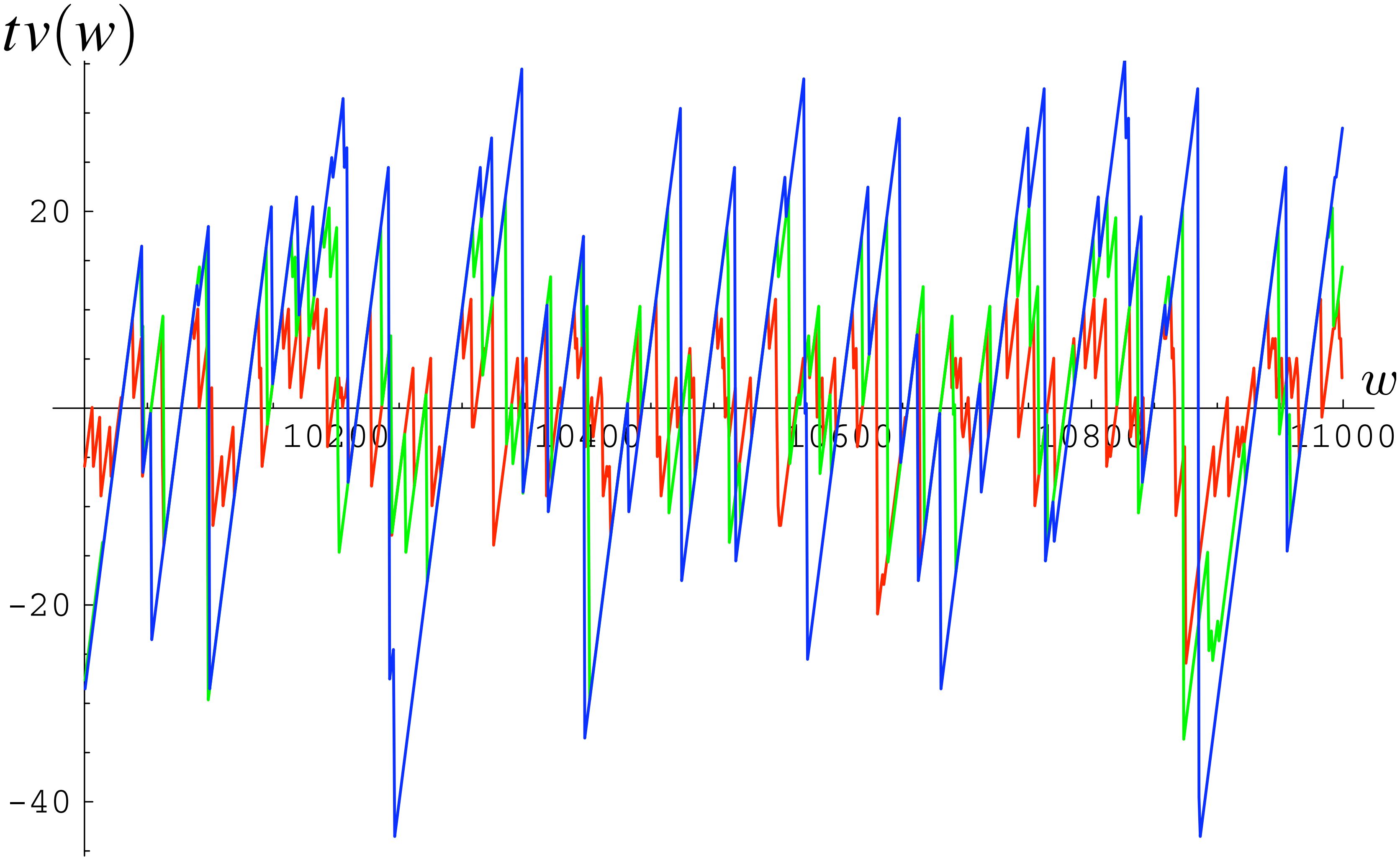
Solution

$$V(\vec{r},t) = \min_{\vec{u}} \left[\frac{1}{2t}(\vec{u}-\vec{r})^2 + V(\vec{u},t=0) \right]$$

$\uparrow$

particle in parabola with curvature $m^2 = \dfrac{1}{t}$

Decaying Burgers



Decaying Burgers with random walk initial condition

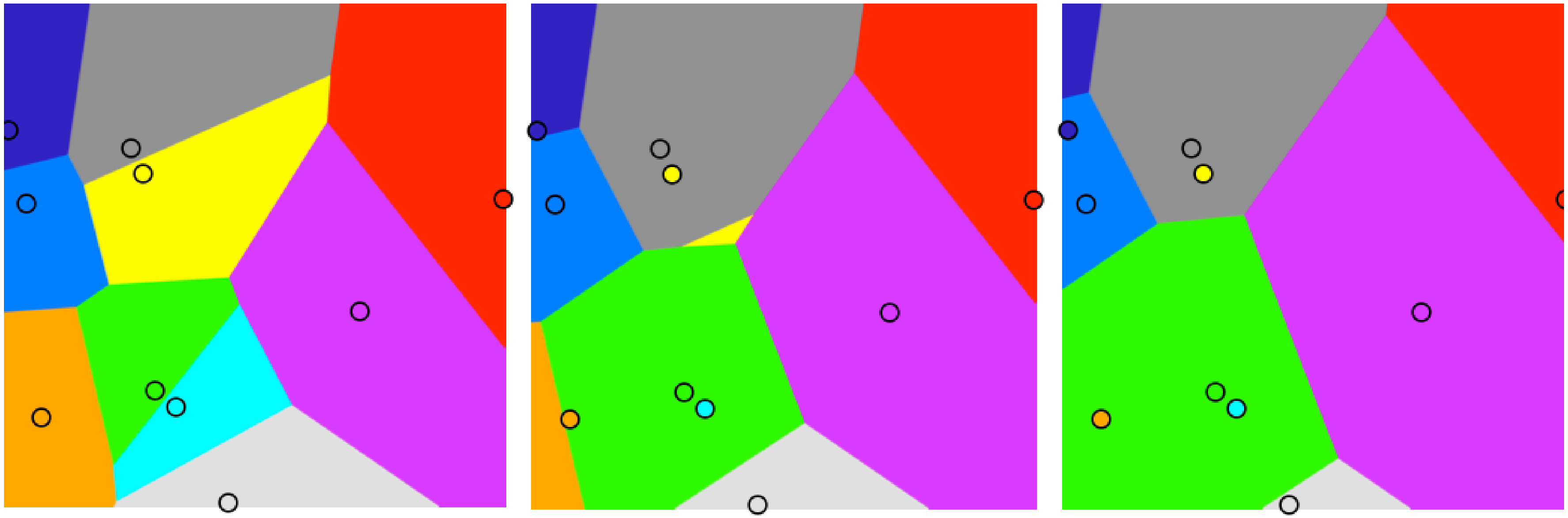
$$d = 1: \qquad \frac{1}{2}\left\langle \left[v(r,t=0) - v(r',t=0)\right]^2 \right\rangle = A|r-r'|$$

$$\Delta(r) = \Delta(0) - A|r-r|$$

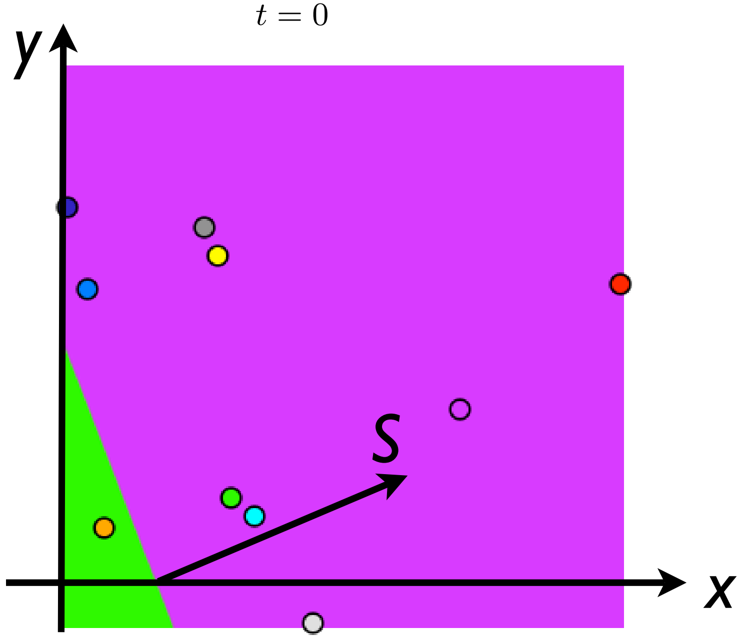
FRG magic: $\Delta(r)$ does not renormalize!

$\Rightarrow$ tree result is exact

$$P(S) = \frac{\mathrm{e}^{-S/S_m}}{S^{3/2}} \qquad\qquad S_m = At^2$$



$t = 0$ $t = 0.045$ $t = 0.1$



$t = 1$

Decaying Burgers in 2d

shocks merge upon
increasing time

Analytical Results for Decaying Burgers in 2 dimensions

?

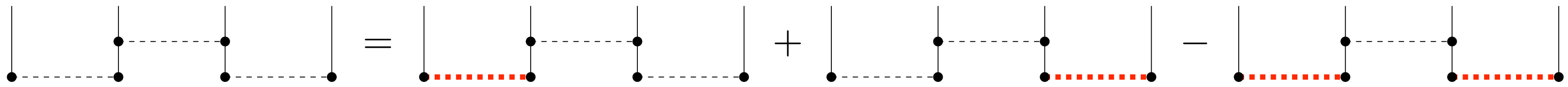
Analytical Results for Decaying Burgers in 2 dimensions

Funtional RG still exact for

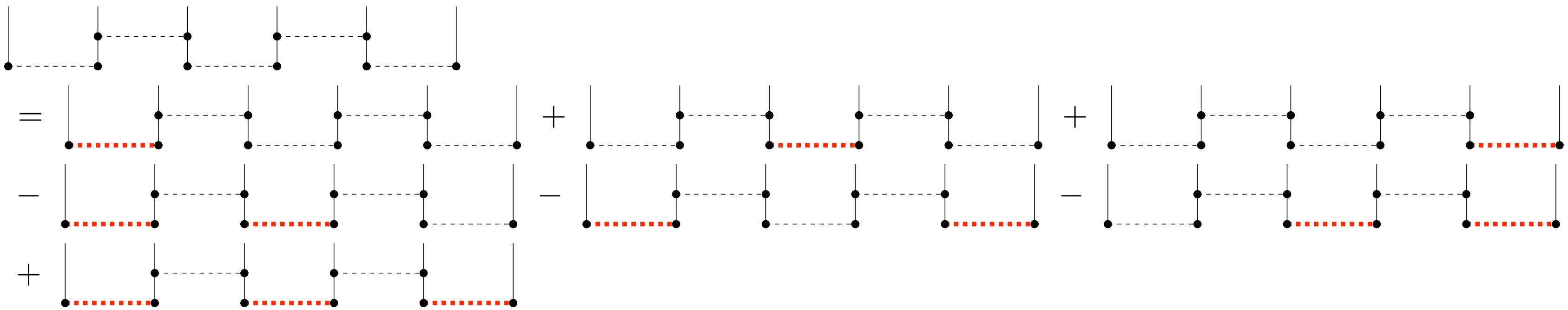
$$\frac{1}{2}\left\langle \left[\vec{v}(\vec{r},t=0)-\vec{v}(\vec{r}',t=0)\right]^2\right\rangle = A|\vec{r}-\vec{r}'|$$

but tree summation becomes difficult, since non-rooted trees contribute

Tree resummation (2): several "roots"



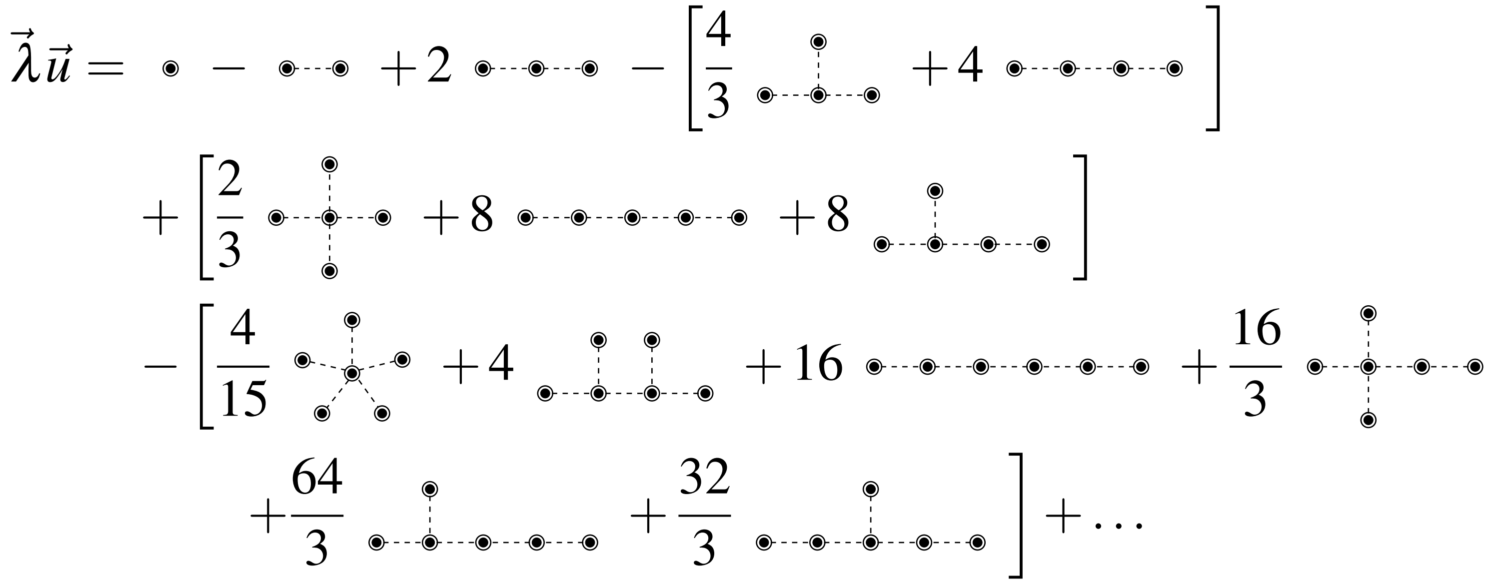
standard recursion relation overcounts - correct



more complicated consitency relation

$$\vec{\lambda}\vec{u} = \lim_{v\to 0}\frac{1}{v}\ln\left(\mathrm{e}^{-v\sum_{i,j} r_{ij}\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}}\mathrm{e}^{\frac{1}{v}Z(\vec{\lambda},\vec{u})}\right)$$

$$\vec{\lambda}\vec{u} = \lim_{\nu \to 0} \frac{1}{\nu} \ln \left(e^{-\nu \sum_{i,j} r_{ij} \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j}} e^{\frac{1}{\nu} Z(\vec{\lambda},\vec{u})} \right)$$



$\Leftrightarrow$

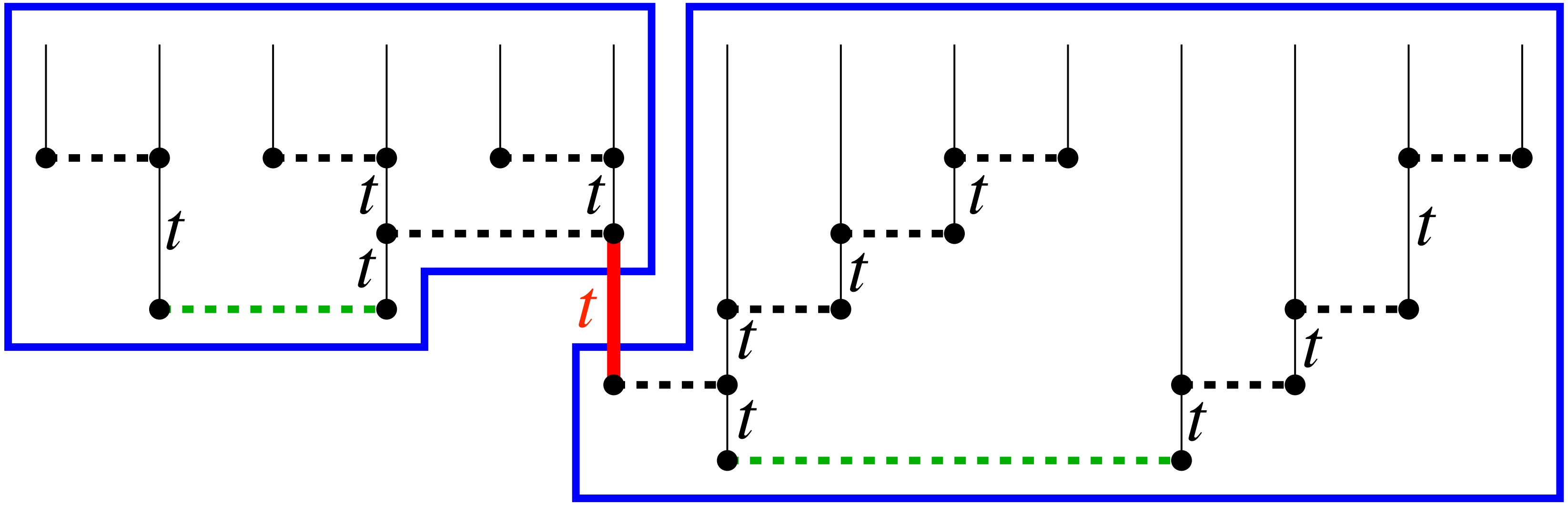
$$\vec{u}\vec{\lambda} = Z_\lambda(y) + \sum_{ij} r_{ij} \partial_i Z_\lambda(y) \partial_j Z_\lambda(y)$$

$\Leftrightarrow$

$$r_{ij} = \frac{1}{2} \left[\frac{u_i u_j}{|\vec{u}|} + \delta_{ij} |\vec{u}| \right] \frac{h'''(0)}{m^4}$$

$$y_i = u_i - 2 r_{ij} \partial_j Z_\lambda(y)$$

Tree resummation (3)



"time" evolution equation

$$\partial_t Z(\vec{\lambda},\vec{u}) = -\frac{\partial}{\partial u} Z(\vec{\lambda},\vec{u}) \frac{\partial}{\partial \lambda} Z(\vec{\lambda},\vec{u})$$

$$Z_{t=0}(\vec{\lambda},\vec{u}) = \vec{\lambda}\cdot[\Delta(0)-\Delta(\vec{u})]\cdot\vec{\lambda}$$

Results

Finite moments

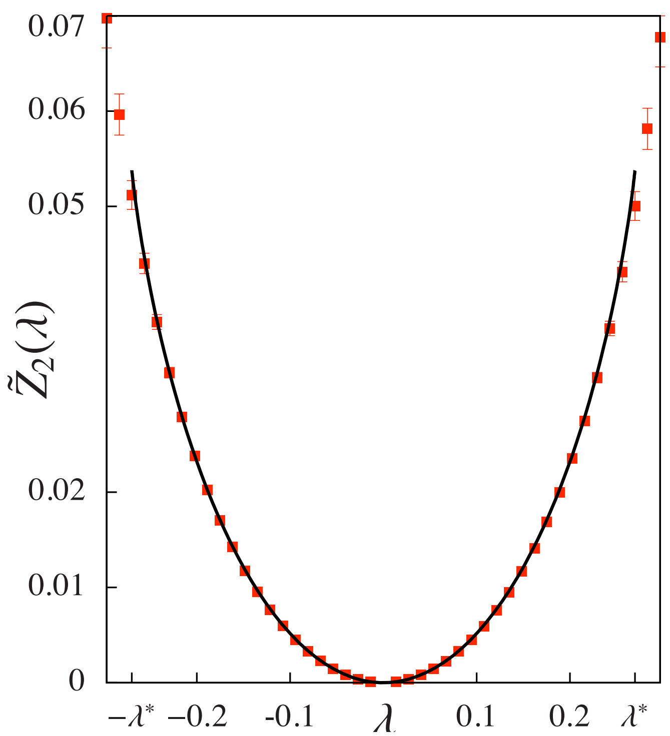
$$\frac{Z(\vec{\lambda},\vec{u})}{|\vec{u}|} = \lambda_1 + \frac{1}{2}\left[\lambda_1^2+\vec{\lambda}^2\right] + 2\lambda_1\vec{\lambda}^2 + \left[\frac{3}{2}(\vec{\lambda}^2)^2 + \frac{9}{2}\vec{\lambda}^2\lambda_1^2 - \lambda_1^4\right] + \left[-\frac{3}{2}\lambda_1^5 + 3\lambda_1^3\vec{\lambda}^2 + \frac{25}{2}\lambda_1(\vec{\lambda}^2)^2\right]$$
$$+\frac{3}{16}\left[13\lambda_1^6 - 93\lambda_1^4\vec{\lambda}^2 + 259\lambda_1^2(\vec{\lambda}^2)^2 + 45(\vec{\lambda}^2)^3\right] + \left[14\lambda_1^7 - 57\lambda_1^5\vec{\lambda}^2 + 72\lambda_1^3(\vec{\lambda}^2)^2 + 103\lambda_1(\vec{\lambda}^2)^3\right]$$
$$+\frac{1}{16}\left[977(\vec{\lambda}^2)^4 + 9017\lambda_1^2(\vec{\lambda}^2)^3 - 3611\lambda_1^4(\vec{\lambda}^2)^2 + 287\lambda_1^6\vec{\lambda}^2 + 194\lambda_1^8\right]$$
$$+\frac{1}{8}\left[7741(\vec{\lambda}^2)^4\lambda_1 + 10644(\vec{\lambda}^2)^3\lambda_1^3 - 10842(\vec{\lambda}^2)^2\lambda_1^5 + 4548(\vec{\lambda}^2)\lambda_1^7 - 651\lambda_1^9\right] + O(\lambda^9)$$

Universal ratio

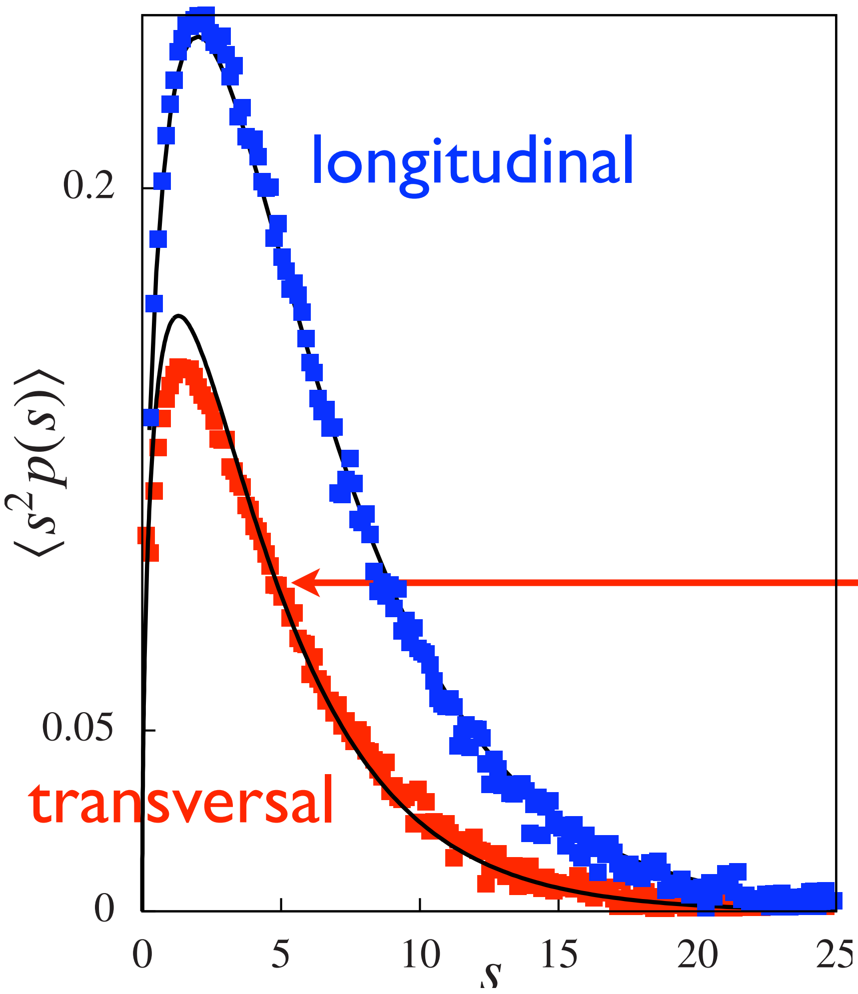
$$\frac{\langle S_x^2\rangle}{\langle S_\perp^2\rangle} = \frac{2}{\langle s_\perp^2\rangle} = \frac{2}{D-1}$$

Generating function (y direction)

$$\lambda(\theta) = \sin\theta\frac{\sqrt{5-\cos(4\theta)}+2}{\left[1-\cos(2\theta)+\sqrt{5-\cos(4\theta)}\right]^2}$$
$$\tilde{Z}_2(\theta) = \frac{\cos\theta}{2}\frac{\sqrt{5-\cos(4\theta)}-2}{1-\cos(2\theta)+\sqrt{5-\cos(4\theta)}}\ .$$

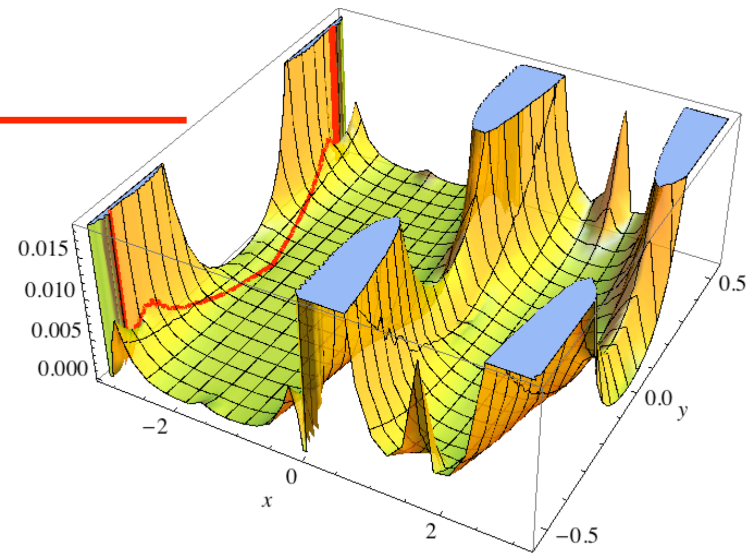


Probability distribution function for shocks

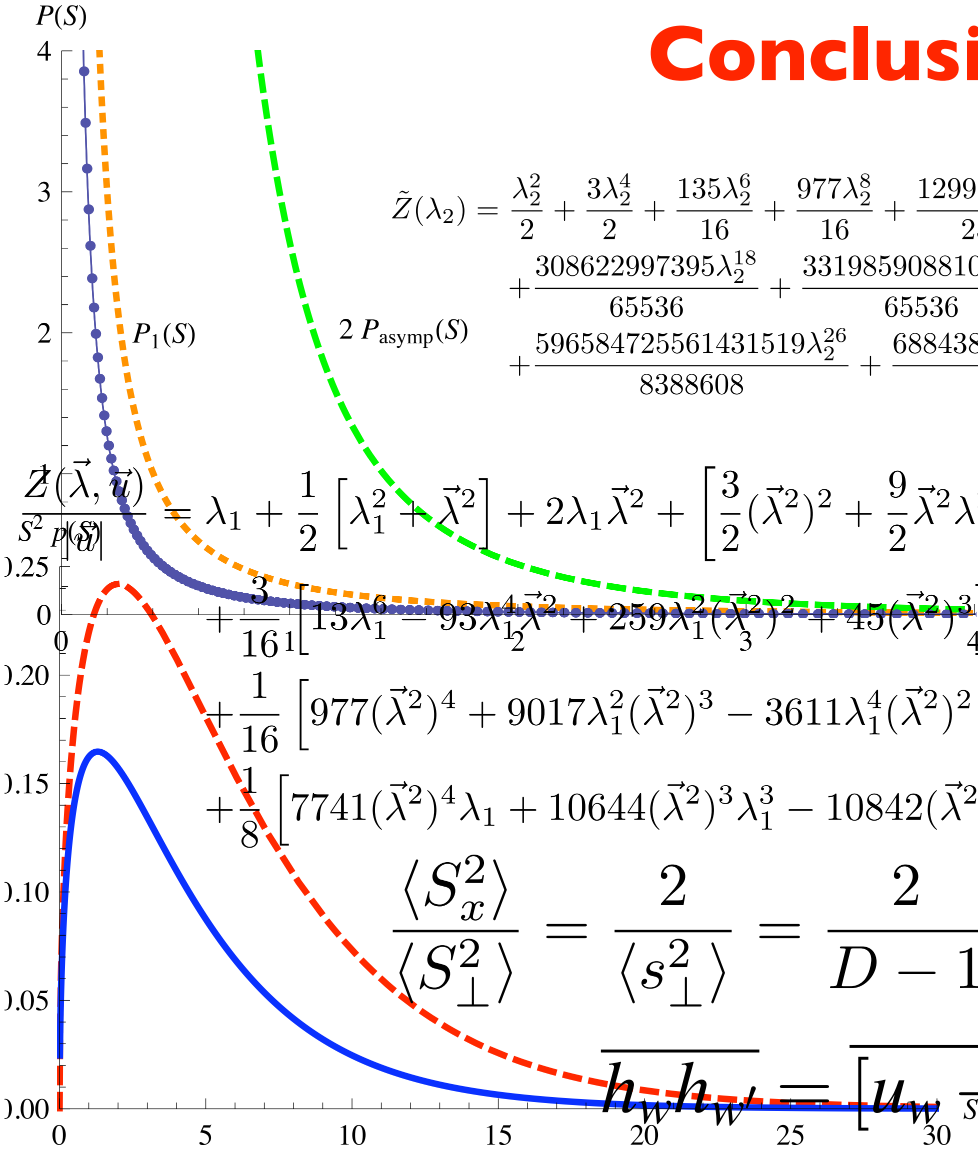


$$s \to \infty$$

$$\frac{1.73040189364e^{-0.269806s}}{s^{5/2}}$$



Conclusions



$$\tilde Z(\lambda_2) = \frac{\lambda_2^2}{2} + \frac{3\lambda_2^4}{2} + \frac{135\lambda_2^6}{16} + \frac{977\lambda_2^8}{16} + \frac{129963\lambda_2^{10}}{256} + \frac{1180377\lambda_2^{12}}{256} + \frac{91202891\lambda_2^{14}}{2048} + \frac{921684933\lambda_2^{16}}{2048} + \frac{308622997395\lambda_2^{18}}{65536} + \frac{3319859088105\lambda_2^{20}}{65536} + \frac{292183179645897\lambda_2^{22}}{524288} + \frac{3274855720092783\lambda_2^{24}}{524288} + \frac{596584725561431519\lambda_2^{26}}{8388608} + \frac{6884384627001260037\lambda_2^{28}}{8388608} + \frac{642973844837708624403\lambda_2^{30}}{67108864} + O\left(\lambda_2^{31}\right)$$

$$\frac{Z(\vec\lambda,\vec u)}{|\vec u|} = \lambda_1 + \frac{1}{2}\left[\lambda_1^2 + \vec\lambda^2\right] + 2\lambda_1\vec\lambda^2 + \left[\frac{3}{2}(\vec\lambda^2)^2 + \frac{9}{2}\vec\lambda^2\lambda_1^2 - \lambda_1^4\right] + \left[-\frac{3}{2}\lambda_1^5 + 3\lambda_1^3\vec\lambda^2 + \frac{25}{2}\lambda_1(\vec\lambda^2)^2\right] + \frac{3}{16}\left[13\lambda_1^6 - 93\lambda_1^4\vec\lambda^2 + 259\lambda_1^2(\vec\lambda^2)^2 + 45(\vec\lambda^2)^3\right] + \left[14\lambda_1^7 - 57\lambda_1^5\vec\lambda^2 + 72\lambda_1^3(\vec\lambda^2)^2 + 103\lambda_1(\vec\lambda^2)^3\right] + \frac{1}{16}\left[977(\vec\lambda^2)^4 + 9017\lambda_1^2(\vec\lambda^2)^3 - 3611\lambda_1^4(\vec\lambda^2)^2 + 287\lambda_1^6\vec\lambda^2 + 194\lambda_1^8\right] + \frac{1}{8}\left[7741(\vec\lambda^2)^4\lambda_1 + 10644(\vec\lambda^2)^3\lambda_1^3 - 10842(\vec\lambda^2)^2\lambda_1^5 + 4548(\vec\lambda^2)\lambda_1^7 - 651\lambda_1^9\right] + O(\lambda^9)$$

$$\frac{\langle S_x^2\rangle}{\langle S_\perp^2\rangle} = \frac{2}{\langle s_\perp^2\rangle} = \frac{2}{D-1} \qquad S_m := \frac{\langle S^2\rangle}{2\langle S\rangle} = \frac{|\Delta'(0^+)|}{m^4}$$

$$\overline{h_w h_{w'}} = \overline{[u_w - w]\,[u_{w'} - w']} = \frac{\Delta(w-w')}{L^d m^4}$$

??? WHERE ARE THE EXPERIMENTS ???